1st Class

Mathematic

الرياضيات

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Chapter One

Consider an arbitrary system of equation in unknown as:

AX = B ………………………………………………………….(1)

$$\left.\begin{array}{l} a_{r1}X_1 + a_{12}X_2 + a_B X_3 + ... + a_{1n}X_n \\ ail\chi_1 + ai_2\chi_2 + ai_3\chi 3 + ... + ain\chi n = b_1 \\ a21\chi_1 + a22\chi 2 + a23\chi 3 + ... + a2n\chi n = b2 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + ... + a_{2n}X_n \\ am1\chi 1 + am2X2 + 9m3\chi 3 + ... + a2n\chi n = bm \\ a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + ... + a_{mn}X_n \end{array}\right\} \text{……………………….(2)}$$

The coefficient of the variables and constant terms can be put in the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a2 & a22 & a2n \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix}_{mxn} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \\ \chi n \end{pmatrix}_{nx1} \begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx1} \text{…………………………………..(3)}$$

Let the form

$$\begin{pmatrix} a_u a_{12} a_{1n} \\ \\ a_{21} a_{22} a_{2n} \\ \\ am_1 am_2 amn \end{pmatrix} = A = (a_{i1}) \text{…………………………………………...(4)}$$

Is called (mxn) matrix and donated this matrix by:

[aij] i = 1, 2,……m and j = 1, 2,…..n.

We say that is an (mxn) matrix or تكملة

The matrix of order (mxn) it has m rows and n columns.

For example the first row is (a_{11}, a12, a_{1n})

And the first column is $\begin{pmatrix} a_{12} \\ a_{11} \\ a_{21} \\ a_{m1} \end{pmatrix}$

(aij) denote the element of matrix. Lying in the i – th row and j – th column, and we call this element as the (i,j) - th element of this matrix

Also $\begin{pmatrix} \chi 1 \\ \chi_2 \\ \\ \chi_n \end{pmatrix}_{nx1}$ is (nx1) [n rows and columns]

$\begin{pmatrix} b1 \\ b2 \\ \\ bm \end{pmatrix}_{mx}$ Is (mx1) [m rows and 1 column]

Sub – Matrix:

Let A be matrix in (4) then the sub-matrix of A is another matrix of A denoted by deleting rows and (or) column of A.

Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Find the sub-matrix of A with order (2×3) any sub-matrix of A denoted by deleting any row of A $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Definition 1.1**:**

Tow (mxn) matrices A = [aij] (mxn) and B = [bij] (mxn) are said to be equal if and only if:

aij = bij for i = 1,2…..m and j = 1,2….n

Thus two matrices are equal if and only if:

i. They have the same dimension, and

ii. All their corresponding elements are equal for example:

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & 0(7) & -2+1 \\ 3 & \frac{20}{4} & 2 \end{bmatrix}$$

Definition 1.2

If A = [aij] mxn and B = [bij] mxn are mxn matrix their sum is the mxn matrix A+B $= [aij + bij]mxn$.

In other words if two matrices have the same dimension, they may be added by addition corresponding elements. For example if:

A = $\begin{pmatrix} 2 & -7 \\ -3 & 4 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} -5 & 0 \\ 1 & 6 \end{pmatrix}$

Then

A+B = $\begin{pmatrix} 3+-5 & -7+0 \\ -3+1 & 4+6 \end{pmatrix} = \begin{pmatrix} -3 & -7 \\ -2 & 10 \end{pmatrix}$

Additions of matrices, like equality of matrices is defined only of matrices have same dimension.

Theorem 1.1**:**

Addition of matrices is commutative and associative, that is if A, B and C are matrices having the same dimension then:

A + B = B + A (commutative)

A + (b + C) = (A + B) + C (associative)

Definition 1.3

The product of a scalar K and an mxn matrix A = [aij] mxn is the nn,Xn matrix KA = [kaij] mXn for example:

$$6\begin{pmatrix} -1 & 0 & 7 \\ 5 & 2 & -11 \end{pmatrix} = \begin{pmatrix} 6(-1) & 6(0) & 6(7) \\ 6(5) & 6(2) & 6(-11) \end{pmatrix} = \begin{pmatrix} -6 & 0 & 2 \\ 30 & 12 & -6 \end{pmatrix}$$

Application of Matrices

Definition 1.4**:**

If A = [aij] mxn is mxn matrix and B = [bjk] nxp an nxp matrix, the product AB is the mxp matrix C = [cik] nxp in which

$$Cik = \sum_{j=1}^{n} aij\, bik$$

Example1: if A = $\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{pmatrix}_{2x3}$ $and\ B = \begin{pmatrix} b11 \\ b21 \\ b22 \end{pmatrix}_{3\times1}$

A B = $\begin{pmatrix} a11\, b11 + a12\, b21 + a13\, b31 \\ a21\, b11 + a22\, b21 + a23\, b31 \end{pmatrix}_{2\times1}$

Example 2: Let A = $\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & -2 \end{pmatrix}_{3\times2}$ $and\ B = \begin{pmatrix} 3 & 1 & 4 & -5 \\ -2 & 0 & 3 & 4 \end{pmatrix}$

A B **=** $\begin{pmatrix} 0 & 2 & 17 & 2 \\ -11 & -1 & 8 & 21 \\ 19 & 5 & 14 & -33 \end{pmatrix}_{3\times4}$

Note 1.1:

1 **–** in general if A and B are two matrices. Then A B may not be equal of

BA. For example A = $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $BA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\therefore AB \neq BA$

2 – if A B is defined then its not necessary that B A must also be defined.

For example. If A is of order (2×3) and B of order (3×1) then clearly A B is define, but B A is not defined.

1.3 Different Types of matrices:

1 – Row Matrix: A matrix which has exactly one row is called row matrix.

For example (1, 2, 3, 4) is row matrix

2 – Column Matrix: A matrix which has exactly one column is called a column matrix for example $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ is a column matrix.

3 – Square Matrix**:** A matrix in which the number of row is equal to the number of columns is called a square matrix for example $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2×2 square matrix.

A matrix (A) (n×n) A is said to be order   or to be an n-square matrix.

4 **-** Null or Zero Matrix**:** A matrix each of whose elements is zero is called null matrix or zero matrix, for example $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a (2×3) null matrix.

5 **–** Diagonal Matrix: the elements aii are called diagonal of a square matrix (a_{11} a_{22} – a_{nn}) constitute its main diagonal A square matrix whose every element other than diagonal elements is zero is called a diagonal matrix for

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6 – Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are scalar matrix

7 – Identity Matrix: A diagonal matrix whose diagonal elements are all equal to 1 (unity) is called identity matrix or (unit matrix). And denoted by in for

Example $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note1.2: if A is (mxn) matrix, it is easily to define that AIn = A and also ImA = A

Ex: Find AI and IA when $A = \begin{pmatrix} 3 & 7 & 2 \\ 1 & -1 & 3 \end{pmatrix}$

Solution**:** IA $\longrightarrow$ $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}_{2\times2}\begin{pmatrix}3 & 7 & 2\\1 & -4 & 3\end{pmatrix}_{2\times3} = \begin{pmatrix}3 & 7 & 2\\1 & -4 & 3\end{pmatrix}_{2\times3}$

And AI = $\begin{pmatrix}3 & 7 & 2\\1 & -4 & 3\end{pmatrix}_{2\times3} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}_{3\times3} = \begin{pmatrix}3 & 7 & 2\\1 & -4 & 3\end{pmatrix}_{2\times3}$

8 – Triangular Matrix**:** A square matrix (aij) whose element aij = 0 whenever $j \langle i$ is Called a lower triangular matrix.simillary y a square matrix (aij) whose element aij = 0 whenever is called an upper Tringular Matrix

For example: $\begin{pmatrix}1 & 0 & 0\\4 & 5 & 0\\7 & 8 & 9\end{pmatrix}$, $\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}$ are lower triangular matrix

And

$\begin{pmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 6\end{pmatrix}$, $\begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}$, are upper triangular

Definition 1.4:

Transpose of matrix

The transpose of an mxn matrix A is the nxm matrix denoted by $A^{\mathbf{T}}$, formed by interchanging the rows and columns of A the ith rows of A is the ith columns in $A^{\mathbf{T}}$**.**

For Example**:** A = $\begin{pmatrix}1 & 2 & 3\\0 & 1 & -1\end{pmatrix}_{2\times3}$ $A^T = \begin{pmatrix}1 & 0\\2 & 1\\3 & -1\end{pmatrix}_{3\times2}$

9 – Symmetric Matrix: A square matrix A such that A = $A^{\mathbf{T}}$ is called symmetric matrix i.e. A is a symmetric matrix if and only if aij = $a_{\mathbf{ji}}$ for all element.

$$\begin{pmatrix}0 & 1 & 2\\1 & 2 & 3\\2 & 3 & 4\end{pmatrix}$$

For Example: (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, (b)

10 – Skew symmetric Matrix: A square matrix A such that A = A^{T} is called that A is skew symmetric matrix. i.e A is skew matrix $\longleftrightarrow$ a_{ji} = -aij for all element of A.

The following are examples of symmetric and skew – symmetric matrices respectively

$$(a)\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix},(b)\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

(a) symmetric

(b) Skew – symmetric.

Note the fact that the main diagonal element of a skew – symmetric matrix must all be Zero

11 – Determinates: To every square matrix that is assigned a specific number called the determinates of the matrix.

(a) Determinates of order one: write det (A) or $|A|$ for detrimental of the matrix A. it is a number assigned to square matrix only.

The determinant of (1×1) matrix (a) is the number a itself det (a) = a.

(c) Determinants of order two: the determinant of the 2×2. matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Is denoted and defined as follows: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Theorem 1.2: determinant of a product of matrices is the product of the determinant of the matrices is the product of the determinant of the matrices det (A B) = det (A). det (B) det (A + B) # det a + det B

(C) Determinates of order three:

(i) the determinant of matrix is defined as follows:

$$\begin{vmatrix} + & - & + \\ a111 & a22 & a23 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} = a11 \begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix} - a12 \begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix} + a13 \begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$$

(ii) Consider the (3×3) matrix $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$

= a11 a22 a33 + a21 a22 a31 + a13 a21 a32

- a31 a22 a13 - a32 a23 a11 -a33 a21 a12.

Show that the diagram papering below where the first two columns are rewritten to the right of the matrix.

Theorem 1.3:

A matrix is invertible if and only if its determinant is <u>not Zero</u> usually a matrix is said to be singular if determinant is zero and non singular it otherwise.

1.5 prosperities of Determinants

(1) det A = det A^T where A^T is the transpose of A.

(2) if any two rows (or two columns) of a determinates are interchanged the value of determinants is multiplied by -1.

(3) if all elements in row (or column) of a square matrix are zero.

Then det (A) = 0

(4) if two parallel column (rows) of square matrix A are equal then det (A) = 0

(5) if all the elements of one row (or one column) of a determinant are multiplied by the same factor K. the value of the new determinant is K times the given det.

Example;

$$\begin{pmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2.3 & 1 \\ 3 & -3.3 & 2 \\ -1 & 4.3 & 3 \end{pmatrix}$$

$$= 3\begin{pmatrix} 4 & 2 & 1 \\ 3 & -3 & 2 \\ -1 & 4 & 3 \end{pmatrix}$$

Example: $\begin{pmatrix} 1 & 0 & 4 \\ -2 & 5 & -8 \\ 3 & 6 & 12 \end{pmatrix} = 4\begin{pmatrix} 1 & 0 & 1 \\ -2 & 5 & -2 \\ 3 & 6 & 3 \end{pmatrix} = 0$

(6) if to each element of a selected row (or column) of a square matrix = k times. The corresponding element of another selected row (or column) is added.

Example: $\begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & +1 \\ 3 & 0 & 2 \end{vmatrix} = -1\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = 2$

2× row (1) + row (3) $\begin{vmatrix} 2 & 0 & 2 \\ 1 & -1 & 1 \\ 7 & 0 & 6 \end{vmatrix} = -1\begin{vmatrix} 2 & 2 \\ 7 & 6 \end{vmatrix} = 2$

(7) if any row or column contain zero elements and only one element not zero then the determinant will reduced by elementary the row and column if the specified element indeterminate.

1.6 Rank of Matrix: we defined the rank of any matrix a that the order of the largest square sub-matrix of a whose determinant not zero (det of sub-matrix ‡ 0)

Example**:** Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ find the rank of A

1 × 9×5 + 2 × 6 × 7 + 3 × 4 × 8 – 3 × 5 × 7 – 1 × 6 × 8 – 2 × 4 × 9 = 0

Since $|A|$ of order 3 Rank ‡ 3

Since $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \neq 0$ the rank ‡ 2

1.7 Minor of matrix**:** Let A = $\begin{pmatrix} a11 & a12 & a1n \\ a21 & a22 & a2n \\ & & \\ an1 & an2 & ann \end{pmatrix}$ (4)

Is the square matrix of order n then the determinant of any square sub-matrix of a with order (n-1) obtained by deleting row and column is called the minor of A and denoted by Mij.

1.8 Cofactor of matrix: Let A be square matrix in (4) with mij which is the minors of its. Then the Cofactor of a defined by Cij = $(-1)^{i+j}$ Mij

Example: Let A = $\begin{pmatrix} -2 & 4 & 1 \\ 4 & 5 & 7 \\ -6 & 1 & 0 \end{pmatrix}$ find the minor and the cofactor of element 7.

Solution: The minor of element 7 is

M23 = det $\begin{pmatrix} -2 & 4 \\ -6 & 1 \end{pmatrix} = \begin{vmatrix} -2 & 4 \\ -6 & 1 \end{vmatrix} = 22$

i.e (denoted by take the square sub-matrix by deleting the second rows and third column in A).

the Cofactor of 7 is

$$C23=(-1)^{2+3}M_{23}=(-1)^{2+3}\begin{vmatrix}-2 & 4\\ -6 & 1\end{vmatrix}=-22$$

1.9 Adjoint of matrix: Let matrix A in (4) then the transposed of matrix of cofactor of this matrix is called adjoint of A, adjoint A = transposed matrix of Cofactor.

The inverse of matrix**:** Let A be square matrix. Then inverse of matrix {Where A is non-singular matrix} denoted by A^{-1} and $A^{-1} = \frac{1}{\det A} adj(A)$

1.0 method to find the inverse of A**:** To find the inverse of matrix we must find the following:

(i) the matrix of minor of elements of A.

(ii) the Cofactor of minor of elements of A

(iii) the adjoint of A .

then $A^{-1} = \frac{1}{|A|} adjA$

Example: let A = $\begin{pmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{pmatrix}$ Find A^{-1}

(1) Minors of A is Mij = $\begin{pmatrix}1 & -10 & -7\\ -7 & 10 & -11\\ 17 & 10 & 1\end{pmatrix}$

(2) Cofactor of A is (-1) Mij = $\begin{pmatrix}1 & 10 & -7\\ 7 & 10 & 11\\ 17 & -10 & 1\end{pmatrix}$

(3) Adj of A = $\begin{pmatrix}1 & 7 & 17\\ 10 & 10 & -10\\ -7 & 11 & 1\end{pmatrix}$.

(4) det = 60

$$A^{-1} = \frac{1}{60}\begin{pmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{pmatrix}.$$

1.11 Properties of Matrix Multiplication:

1 – (KA) B = K (AB) = A (KB)   K is any number

2 – A (BC) = (AB) C

3 – (A + B) C = AC + BC

4 – C (A + B) = CA + CB

5 – AB ‡ BA (in general)

For example: Let A = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

A B = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

B A = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

A B = ‡ B A

6 – A B = 0 but not necessarily A = 0 or B = 0

For Example: A = $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A \neq 0, B \neq 0$$

But

A B = 0

7 - $\begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} = C\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8 – A I = I A = A   where I is identity matrix

9 – $(A\ B)^{T} = B^{T}\ A^{T}$

10 – $A^{-1}\ A = A.A^{-1} = I$

1.12 Cramer's Rule

Let the system of linear question as

$$\left.\begin{array}{l} a_{11}\ \chi_1 + a_{12}\ \chi_2 = b1 \\ a_{21}\ \chi_1 + a_{22}\ \chi_2 = b2 \end{array}\right\} \rightarrow (i)$$

The system (i) can put in the form:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{11} & a_{22} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \rightarrow (ii)$$

If D = $\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix} \neq 0$

Then the system (ii) has a unique solution, and Cramer's rule state that it may be found from the formulas:

$$\chi_1 = \frac{\begin{vmatrix} b1 & a12 \\ b2 & a22 \end{vmatrix}}{D}\ ,\ X_2 = \frac{\begin{vmatrix} a11 & b1 \\ a21 & b2 \end{vmatrix}}{D}$$

Example: solve the system

$3X_1 - \chi_2 = 9$

$X_1 + 2X_2 = -4$

So, the system can put in the form

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$$

D = $\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7$, $\chi_1 = \dfrac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{D} = \dfrac{14}{7} = 2$

$$\chi_2 = \frac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{D} = \frac{21}{7} = 3$$

Let the following system in the unknowns:

$$\begin{aligned} a11\ \chi_1 + a11\ \chi_2 + a11\,3X_3 &= b1 \\ a21\ \chi_1 + a22\ \chi_2 + a23\,X_3 &= b2 \\ a31\ \chi_1 + a32\,X_2 + a33\ \chi_3 &= b3 \end{aligned}$$

The system (I) can be put in the form:

$$\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \begin{pmatrix} b1 \\ b2 \\ b3 \end{pmatrix} \qquad \text{(II)}$$

If D = $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} \neq 0$

The system has a unique solution, given by Cramer's rule:

$$\chi_1 = \frac{1}{D}\begin{vmatrix} b1 & a12 & a13 \\ b2 & a22 & a23 \\ b3 & a32 & a33 \end{vmatrix},\ X_2 = \frac{1}{D}\begin{vmatrix} a11 & b1 & a13 \\ a21 & b2 & a23 \\ a31 & b3 & a33 \end{vmatrix}\ X_3 = \frac{1}{D}\begin{vmatrix} a11 & a12 & b1 \\ a21 & a22 & b2 \\ a31 & a32 & b3 \end{vmatrix}$$

Example: solve the system

$$\begin{aligned} X_1 + 3X_2 - 2X_3 &= 11 \\ 4X_1 - 2X_2 + X_3 &= -15 \\ 3X_1 + 4X_2 - X_3 &= 3 \end{aligned}$$

By cramer's rule.

The system (1) become $\begin{pmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{pmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{pmatrix} 11 \\ -15 \\ 3 \end{pmatrix}$

Since D = det = $\begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$

Cramer's rule gives the solution:

$$\chi_1 = \frac{\begin{vmatrix} 11 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = \frac{50}{-25} = -2$$

$$\chi_2 = \frac{\begin{vmatrix} 1 & 11 & -2 \\ 4 & -15 & 1 \\ 3 & 3 & -1 \end{vmatrix}}{-25} = \frac{-25}{-25} = 1$$

$$\chi_3 = \frac{\begin{vmatrix} 1 & 3 & 11 \\ 4 & -2 & -15 \end{vmatrix}}{-25} = \frac{125}{-25} = -5$$

Chapter Two

Function

Numbers:

1 – N = set of natural numbers

N = {1, 2, 3, 4……}

2 – I = set of integers

= {……., -3, -2, -1, 0, 1, 2, 3…}

Note that: NCI

3 – A = set of rational numbers

$= \left(\chi : \chi = \frac{\rho}{q} \ \rho \, and \, q \, are \, integers \, q \neq 03 \right]$

Ex: $\frac{3}{2}, -\frac{4}{5}, \frac{3}{1}, \frac{-7}{1}$

Note that: ICA

4 – B = set of irrational numbers

= $\{X : X \text{ is not a rational number}\}$

Ex: $\sqrt{2}$, $\sqrt{3}$, $-\sqrt{7}$

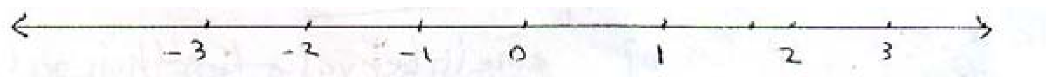
5 – R: set of real numbers

= set of all rational and irrational numbers

Note that

R = AUB

Note: the set of real numbers is represented by a line called a line of numbers:



(ii) NCR, ICR, ACR, BCR

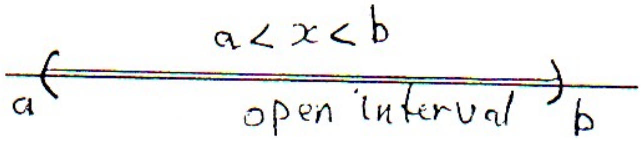
Intervals

The set of values that a variable χ may take on is called the domain of χ. The domains of the variables in many applications of calculus are intervals like those shows below.

- **open intervals**

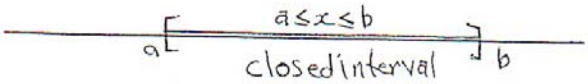
is the set of all real numbers that lie strictly between two fixed numbers a and b:

In symbols	**In words**
$a \langle \chi \langle b \; or \; (q, b)$	The open interval a b



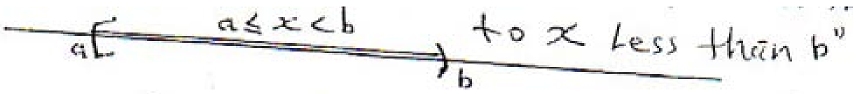
- Closed Intervals contain both endpoints:

In symbols	**In words**
$a \leq \chi \leq b \quad or \quad [a,b]$	the closed interval a b



- Half – open intervals contain one but not both end points:

In symbols:	**in wards**
$a \leq \chi \langle b$ or [a,b]	‘ the interval a less than or equal
$a \leq \chi c b$	To χ less than b



$a \langle \chi \leq b$ or [a,b] the interval a less than χ less than or equal b

Ex. find the domain of

1 – Y = $\sqrt{1-X^2}$

The domain of χ is the closed interval

$1 - \leq \chi \leq 1$

2 – Y = $\frac{1}{\sqrt{1-X^2}}$

The domain for χ is open interval

$-1 \langle \chi \langle 1$ because $\frac{1}{0}$ is not defined

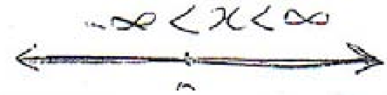
B - y = $\sqrt{\frac{1}{X}-1}$

$1/X - 1 \geq 0 \quad or \quad \frac{1}{X} \geq 1$

The domain for χ is the half – open $0 < \chi \le 1$

Ex: **the equation** **the domain**

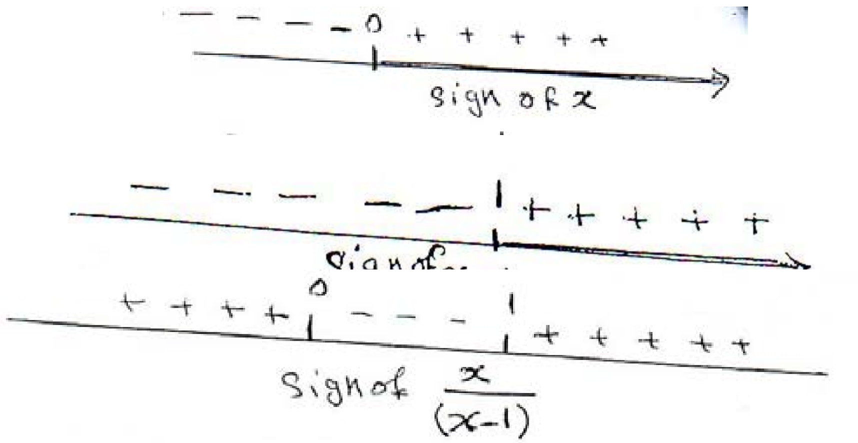
Y = χ^2



Y = $\sqrt{\chi}$

Y = $\frac{1}{\sqrt{4-X}}$

$y = \sqrt{\frac{\chi}{(\chi-1)}}$



The domain for χ is $X \le 0 \cup X > 1$

Definition: A function, say f is a relation between the elements of two sets say A and B such that for every $\chi \in A$ there exists one and only one $Y \in B$ with Y = F(X).

The set A which contain the values of χ is called the domain of function F.

The set B which contains the values of Y corresponding to the values of χ is called the range of the function F. χ is called the independer variable of the function F, while Y is called the dependant variable of F.

Note:

1 – Some times the domain is denoted by DF and the range by RF.

2 – Y is called the image of χ.

Example: Let the domain of χ be the set {0,1,2,3,4}. Assign to each value of χ the number Y = χ^2. The function so defined is the set of pairs, { (0,0), (1,1), (2,4), (3,9), (4,16) }.

Example: Let the domain of χ be the closed interval $-2 \le \chi \le 2$. Assign to each value of χ the number y = χ^2.

The set of order pairs (χ, y) such that $-2 \le \chi \le 2$

And y = χ^2 is a function.

Note: Now can describe function by two things:

1 – the domain of the first variable χ.

2 – the rule or condition that the pairs (χ, y) must satistfy to belong to the function.

Example:

The function that pairs with each value of χ diffrent from 2 the number

$$\frac{\chi}{\chi - 2}$$

$$y = f(\chi) = \frac{\chi}{\chi - 2} \quad \chi \ne 2$$

Note 2: Let $f(\chi)$ and $g(\chi)$ be two function.

1 - $(f \pm g)(\chi) = f(\chi) \pm g(\chi)$

2 - $(f.g)(\chi) = f(\chi) \,.\, g(\chi)$

3 - $(\frac{f}{g})\,(\chi) = \frac{f(\chi)}{g(\chi)}$ if g $(\chi) \ne 0$

Example: Let $f(\chi) = \chi + 2, g(\chi) = \sqrt{\chi - 3}$ evaluate

$$f \pm g \,,\, f.g \; and \; \frac{f}{g}$$

So: $(f \pm g)(\chi) = f(\chi) \pm g(\chi) = \chi + 2 \pm (\sqrt{\chi - 3}$

$(f.g)(\chi) = f(\chi) \cdot g(\chi) = (\chi + 2)(\sqrt{\chi - 3}$

$(\frac{f}{g})(\chi) = \frac{f(\chi)}{g(\chi)} = \frac{\chi + 2}{\sqrt{\chi - 3}} \quad \{X : X \rangle 3\}$

Composition of Function:

Let $f(\chi)$ and $g(\chi)$ be two functions

We define: $(fog)(\chi) = f(g(\chi))$

Example: Let $f(\chi) = \chi^2,\ g(\chi) = \chi - 7$ evaluate fog and gof

So: $(fog)(\chi) = f[g(x)] = f(\chi - 7) = (\chi - 7)^2$

$(gof)(\chi) = g[f(\chi)] = g(\chi^2) = \chi^2 - 7$

$\therefore\ fog \neq gof$

Inverse Function

Given a function F with domain A and the range B.

The inverse function of f written f, is a function with domain B and range A such that for every $y \in B$ there exists only $\chi \in A$ with $\chi = f^{-1}(y)$.

Note that: $f^{-1} \neq \frac{1}{f}$

Polynomials: A polynomial of degree n with independent variable, written $\mathbf{f_n(x)}$ or simply $f(\chi)$ is an expression of the form:

$fn(\chi) = q_o + a_1\chi + a_2X^2 + \dots\dots\dots + anX^n$ ....(*)

Where $q_0, a1, \dots\dots, an$ are constant (numbers).

The degree of polynomial in equation (*) is n (the highest power of equation)

Example:

(i) f (χ) = 5X polynomial of degree one.

(ii) f $(\chi) = 3X^5 - 2X + 7$ polynomial of degree five.

(iii) F (χ) = 8 polynomial of degree Zero.

Notes:

The value of χ which make the polynomial f (χ) = 0 are called the roots of the equation $(f(\chi) = 0)$

Example: (χ) = 2 is the root of the polynomial

F (χ) = $\chi^2 - \chi - 2$

Since f (2) = 0

Example: F (χ) Linear function if

F (χ) = $a\chi + b$.

Even Function:

F (χ) is even if f (-x) = F (x)

Example: 1 - F (χ) = $(\chi)^2$ is even since f$(-\chi)$ = $(-\chi)^2$ = $(\chi)^2$ = f (χ)

2 - F (χ) = cos (χ) is even because f$(-\chi)$ = cos $(-\chi)$ = cos (χ) = f (χ)

Odd Function:

If $f(-\chi) = -f(\chi)$ the function is called odd.

Example: 1 - $f(\chi) = \chi^3$ is odd since $f(-\chi) = -\chi 3 = -f(\chi)$

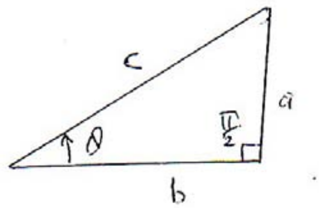
2 - $f(\chi) = Sin(-\chi) = -Sin\, X = -f(\chi)$.

Trigonometric Function:

1 – Sin φ = $\frac{a}{c}$

2 – Cos $\varphi = \frac{b}{c}$

3 – tan $\varphi = \frac{a}{b}$



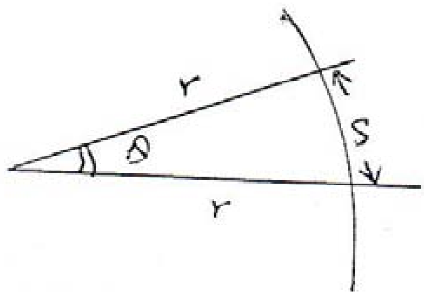
4 – Cotan $\varphi = \frac{1}{\tan\varphi} = \frac{b}{a}$

5 – Sec $\varphi = \frac{1}{\cos\varphi} = \frac{c}{b}$

6- CSC $\varphi = \frac{1}{Sin\varphi} = \frac{c}{a}$

Relation ships between degrees and radians

φ In radius **=** $\frac{s}{r}$



$360^{\mathbf{o}} = \frac{2\pi r}{r}$

$= 2\pi\, radius$

$1^{\mathbf{o}} = \frac{\pi}{180}$ radius = 0.0174 radian

1 radian = $\frac{180}{\pi}\deg ree = 57.29578^{o}$

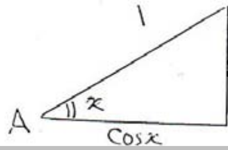
$\left(\frac{360}{2\pi}\right) = 1 radian = 57^{o}.18$

$180^{\mathbf{o}} = \pi$ radians = 3.14159 – radians

$1^{\mathbf{o}} = \frac{2\pi}{360} = \frac{\pi}{180} \approx 0.001754$ radians

$\tan\chi = \frac{\sin\chi}{\cos\chi}$

$Cot\chi = \frac{Cos\chi}{Sin\chi} = \frac{1}{\tan\chi}$



C

SinX

B

$$Sec\,\chi = \frac{1}{\cos\chi}$$

$$Csc\;\chi = \frac{1}{\sin\chi}$$

$$Cos^2\,\chi + Sin^2\,\chi = 1$$

$$\tan^2\chi + 1 = Sec^2\chi$$

$$Cot^2\,\chi + 1 = Csc^2\,X$$

$$Sin(\chi \pm y) = Sin \times Cosy \;\mu\; Cos \times Sin\,y$$

$$Cos(\chi \pm y) = Cos \times Cosy \pm Sin \times Siny$$

$$\tan(\chi \pm y) = \frac{\tan x \pm \tan y}{1 \;\mu \tan x \tan y}$$

1 – $SinA + Sin\,B = 2\,Sin\,\frac{A+B}{2}\,Cos\,\frac{A-B}{2}$

2 - $Sin\,A - Sin\;B = 2\,Cos\,\frac{A+B}{2}\;Sin\,\frac{A-B}{2}$

3 – $Cos\,A + Cos\;b = 2\,Cos\,\frac{A+B}{2}\;Cos\,\frac{A-B}{2}$

4 - $Cos\,A - Cos\,B = 2\,Sin\,\frac{A+B}{2}\;Sin\,\frac{A-B}{2}$

$$Sin\,2\,X = 2\,Sin\;X\;CosX$$

$$Cos^2 = Cos^2 X - Sin^2 X$$

$$= 1 - 2\,Sin^2 X$$

$$= 2Cos^2 X - 1$$

$$Cos^2 x = \frac{1 + Cos^2 x}{2}$$

$$Sin^2\,x = \frac{1 - Cos^2 x}{2}$$

$$Sin(\varphi + 2\pi) = Sin\,\varphi$$

$$Cos(\varphi + 2\pi) = Cos\,\varphi$$

$$\tan(\varphi + \pi) = \tan\varphi$$

Degree	O^o	30^o	45^o	60^o	90^o	180^o	270^o	360^o
θ radius	O	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin θ	O	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{2}}$	1	O	-1	O
Cos θ	1	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	O	-1	O	1
tan θ	O	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$				

$$Cos(\varphi + 2n\pi) = Cos\varphi$$
$$Sin(\varphi + 2n\pi) = Sin\varphi$$
$$Cos(-\varphi) = Cos\varphi$$
$$Sin(-\varphi) = -Sin\varphi$$
$$Cos(\frac{\pi}{2} + \varphi) = Cos\varphi$$
$$\tan(\pi - \varphi) = -\tan\varphi$$
$$\tan(\frac{\pi}{2} + \varphi) = -Cot\varphi$$

Graphs:

The set of points in the plane whose coordinate pairs are also the ordered pairs of function is called the graph of function.

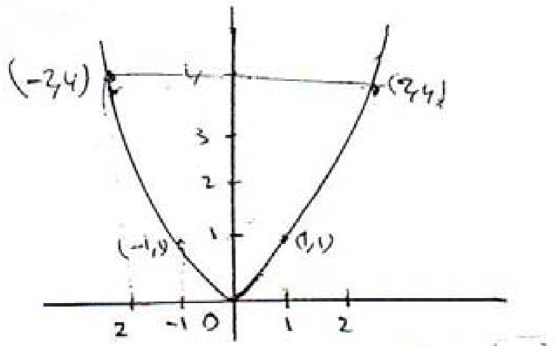
Example: Graph a function we carry out three steps $y = \chi^2, -2 \le x \le 2$

1 – Make a table of pairs from the function as

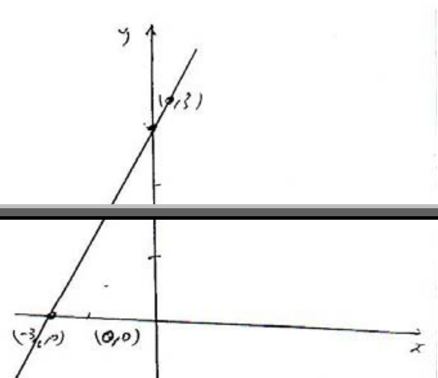
χ	$y = \chi^2$	(χ, y)
-2	4	(2,4)
-1	1	(-1,1)
0	0	(0,0)
+1	1	(1,1)
2	4	(2,4)

2 – Plot enough of the corresponding points to learn the shape of the graph. Add more pairs to the table if necessary.

3 – Complete the sketch by connecting the points.



Example: $y = 2\chi + 3$



X	$y = 2X + 3$	(X,y)
O	3	(0,3)
$-\frac{3}{2}$	O	$(\frac{-3}{2},0)$

Absolute Value:

We define the absolute value function $y = |\chi|$, the function assign every negative number to non-negative, which corresponding points.

The absolute values of X:

$$|X| = \sqrt{X^2} = \begin{cases} \chi & if X \geq 0 \\ -\chi & if X \langle 0 \end{cases}$$

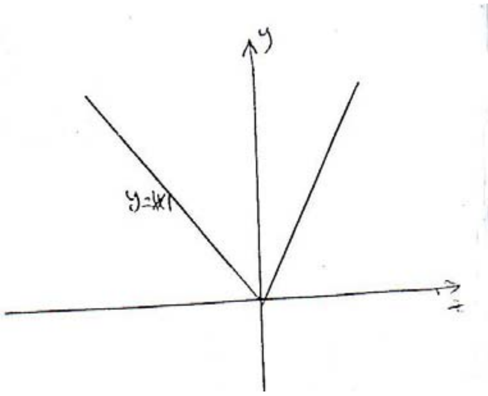
Then:

1 - $|a.b| = |a|,|b|$

2 - $|a + b| \leq |a| + |b|$

3 - $|a| \leq C \Leftrightarrow -C \leq a \leq C$

$y = f(\chi) = \chi$



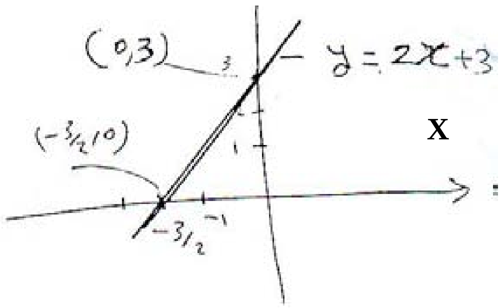
X	y	(X,y)
-1	1	(-1,1)
0	0	(0,0)
1	1	(1,1)

Example 2:

$y = f(\chi) = a\chi + b$

$y = f(\chi) = 2\chi + 3$

X	**y**	**(X,y)**
O	3	(0,3)
$-\frac{3}{2}$	0	$(-\frac{3}{2}, 0)$



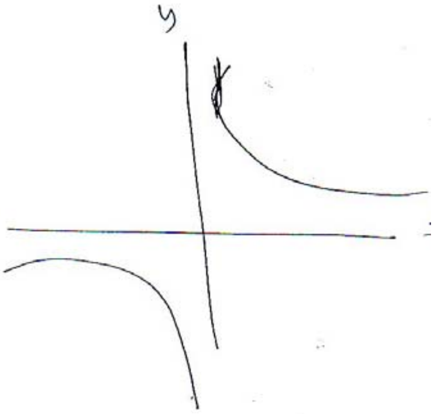
X

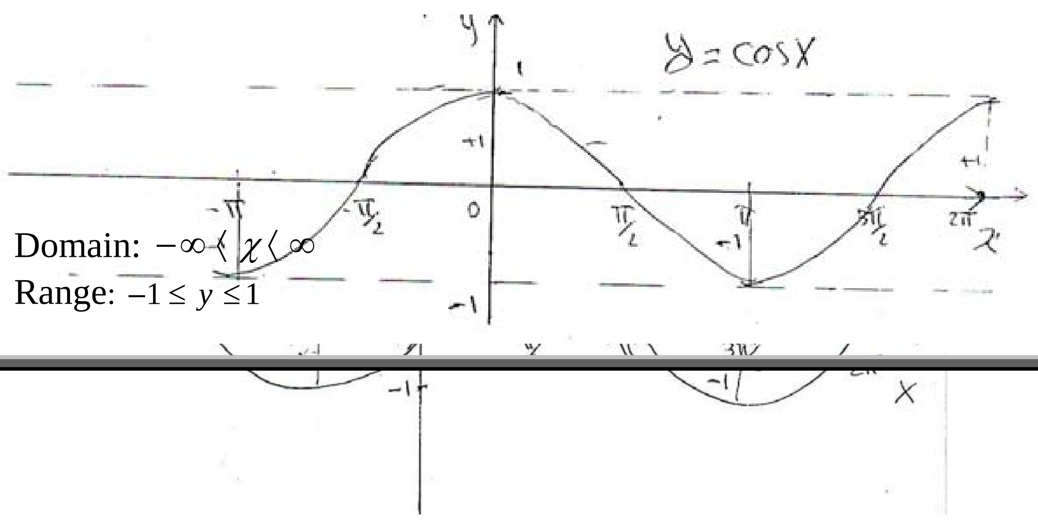
Example 3:

$y = f(\chi) = \frac{1}{\chi}$

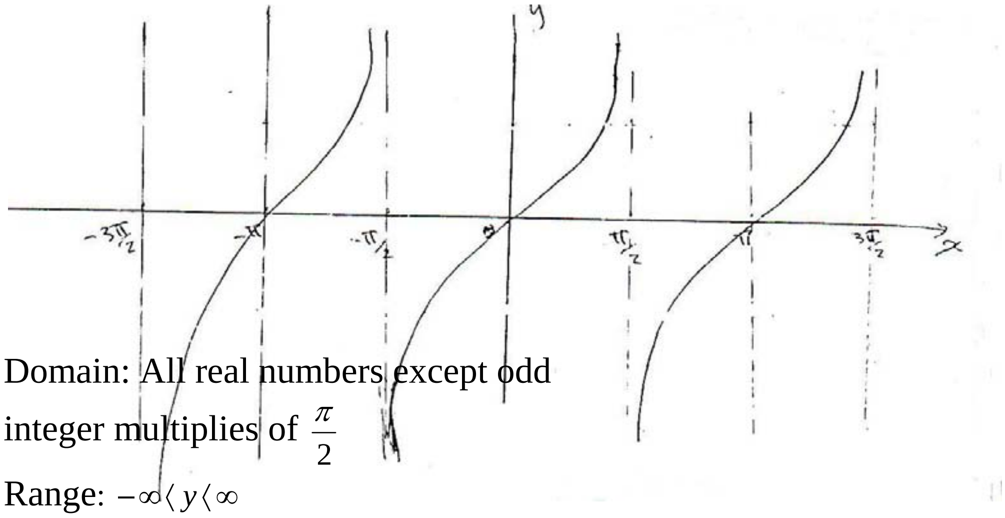
X	y	(X,y)
1	1	(1,1)
-1	-1	(-1,1)
$\frac{1}{2}$	2	$(\frac{1}{2},2)$
$\frac{1}{3}$	3	$(\frac{1}{3},3)$
2	$\frac{1}{2}$	$(2,\frac{1}{2})$
3	$\frac{1}{3}$	$(3,\frac{1}{3})$

Domain: $-\infty \langle \chi \langle \infty$
Range: $-1 \le y \le 1$



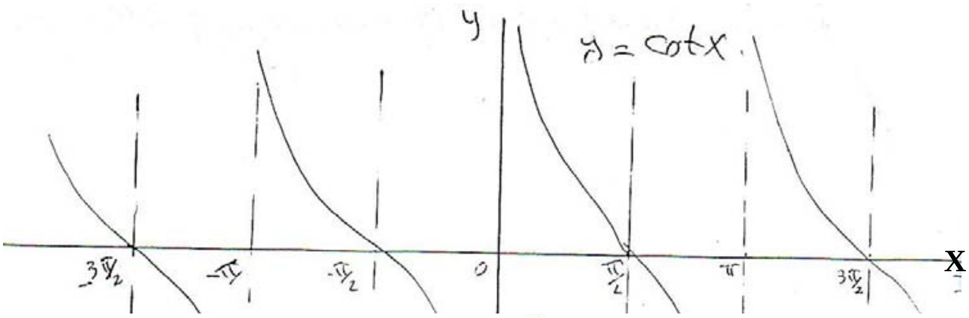


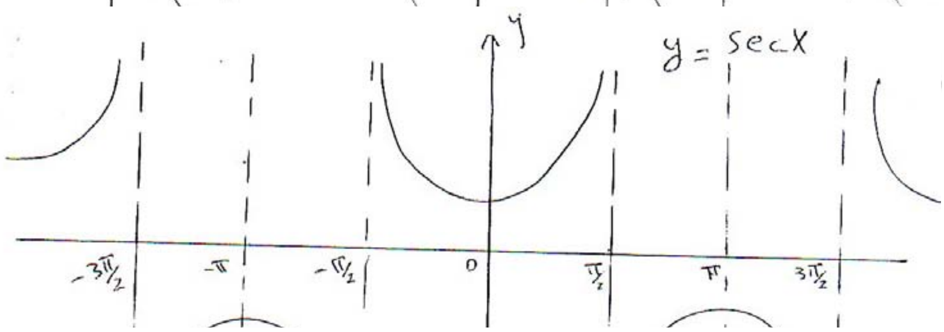
Domain: $-\infty \langle \chi \langle \infty$
Range: $-1 \le y \le 1$

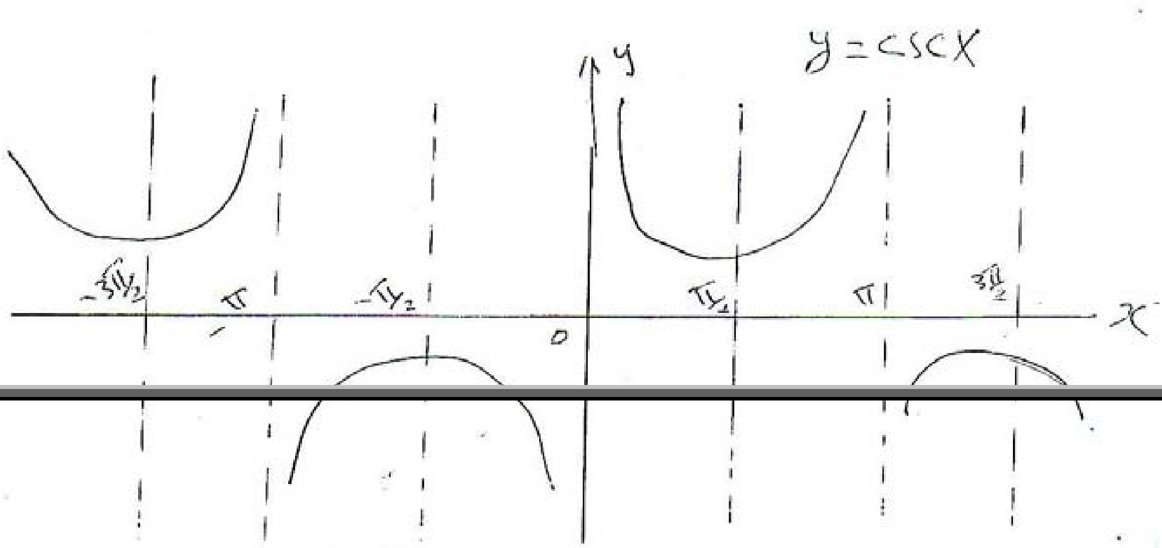


Domain: All real numbers except odd integer multiplies of $\frac{\pi}{2}$

Range: $-\infty \langle y \langle \infty$







Limits:

We say that L is a right hand limit for $f(\chi)$ when X approaches C for the right, written

$$\lim_{X \to C^{+}} f(\chi) = L.$$

Similary, L is the left – hand limit for $f(\chi)$ when X approaches C for the left, written

$$\lim_{X \to C^{-}} = f(\chi) = L,$$

Then $\lim_{X \to C} = f(\chi) = L,$

If and.only if $\lim_{\chi \to C^{+}} f(X) = \lim_{\chi \to C^{-}} f(x)$

Example:

$$\lim_{\chi \to 1} \frac{\chi \pm 1}{\chi - 1} = \lim_{\chi \to 1} \frac{(\chi-1)(\chi+1)}{(\chi-1)}$$

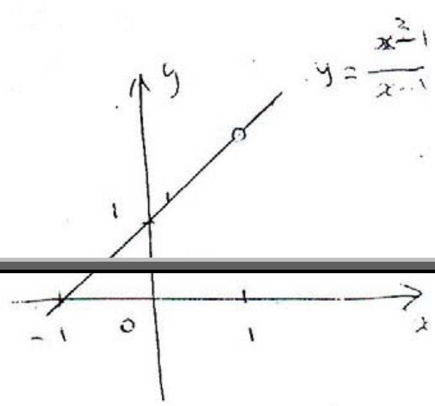
$$\lim_{\chi \to 1} \times + 1 = 1 + 1 = 2$$

Theorem 1

If $\lim_{\chi \to c} f(\chi) = L_1, \lim_{\chi \to c} g(\chi) = L_2$

Then

1 - $\lim_{\chi \to c} [\, f(\chi) \pm g(\chi)] = L1 \pm L2$



2 – Lim $[f(\chi)\,g(\chi)] = L1 \cdot L2$

3 - $Lim\left[\frac{f(\chi)}{g(\chi)}\right] = \frac{L1}{L2}\ if\ L2 \neq 0$

4 - $\underset{\chi \to c}{Lim}\,[K\ f(\chi)] = KL_1$ Where K is a constant

Theorem 2

1 - $Lim\,K = K$, K is constant

2 - $\underset{\chi \to c}{Lim}[a0 + a1\chi_1 + a2\chi^2 + ...+ an\chi^n] = a_0 + a_1c + a_2c^2 ....anc^n$

3 - $\underset{\chi \to 0}{Lim}\, smx = 0$

4 - $\underset{\chi \to 0}{Lim}\ Cos\,\chi = 1$

5 - $\underset{\chi \to 0}{Lim}\frac{Sin\,\chi}{X} = 1$

Example: Evaluate

1 - $\underset{\chi \to 2}{Lim}(4\chi^2) = 4\underset{\chi \to 2}{Lim}\,\chi^2 = 4(2)^2 = 16$

2 - $\underset{x \to 2}{Lim}(\chi^2 - 9) = 4 - 9 = 5$

$3 - Lim\frac{\chi^3 - 8}{\chi^2 - 4} = \frac{(\chi - 2)(\chi^2 + 2\chi + 4)}{(\chi - 2)(\chi + 2)} = \frac{\chi^2 + 2\chi + 4}{\chi + 2}$

$= \frac{12}{4} = 3$

$4 - \underset{\chi \to 0}{Lim}\frac{Sin\chi}{\chi} = 3\underset{3\chi \to 0}{Lim}\frac{Sin3\chi}{3\chi} = 3(1) = 3$

$$5 - \lim_{\chi \to 0} \frac{\tan x}{\chi} = \frac{\frac{Sin\,\chi}{Cos\,\chi}}{\chi} = \lim_{X \to 0} \left[\frac{Sin\,\chi}{Cos} \cdot \frac{1}{\chi} \right]$$

$$= \lim_{X \to 0} \left[\frac{Sin\,\chi}{\chi} \cdot \frac{1}{Cos\,\chi} \right]$$

$$= \left[\lim_{\chi \to 0} \frac{Sin\,\chi}{\chi} \right] \cdot \left[\lim_{\chi \to 0} \frac{1}{Cos\,\chi} \right] = (1)\,(1) = 1$$

Infinity as Limits

Evaluate:

1 - $\lim_{X \to 0^+} \frac{1}{X} = \infty$ (2) $\lim_{\chi \to 0^-} \frac{1}{X} = -\infty$

3 - $\lim_{x \to \infty} \frac{1}{X} = 0$

4 - $\lim_{x \to \infty} \frac{2\chi 2 - \chi + 3}{3\chi^2 - 5} = \lim_{x \to \infty} \frac{\frac{2x2}{X2}}{\frac{3\chi^2}{\chi^2} - \frac{3}{X^2}}$

Theorem

If $f(\chi) \le g(\chi) \le h(X)$ and $Lim\, f(\chi) = Lim\, h(X) = L$ then

L is the limit of g(x)

Example: Evaluate

1 - $\lim_{X \to \infty} \frac{Sin\,X}{X}$

2 - $\lim_{X \to \infty} \chi \, Sin(\frac{1}{X})$

Continuity

Definition: A function f is said to be continuous at $\chi = C$ provided the following conditions are satisfied:

1 $f(C)$ is defined

2 $\lim_{\chi \to C} f(x)$ exists

3 $\lim_{x \to C} f(x) = f(C)$

Theorem

Any Polynomial

1 $P(\chi) = a_0 + a_1\chi + a_2 X^2 + ... + an\,X^4$ (an $\neq 0$)

Is continuous for all χ

2 $R(\chi) = \frac{a_0 + a1X + a_2 X^2 + ... + anX^4}{bo + b_1 X + b_2 X^2 .... + bnX^4}$ $(an \neq 0, bn \neq 0)$

Is continuous at every point of its domain of definition that is at every point where its denominator isd not zero

3 Each of the igonometric function SinX, CosX, tanX, CotX SecX, and CscX, is continuous at every point of its domain of definition.

Example1

$\lim_{X \to \pi} (Cos^2 X + Cos X + 1)$

Solution

$Lim(Cos^2 X + CosX + 1) = (Cos^2 \pi + Cos\,\pi + 1)$

$= (-1)^2 - 1 + 1) = 1$

Example2

$f(\chi) = \frac{|X|}{X}$ Discontinuous at $\chi = 0$

$Lim \frac{|X|}{X} = 1$

$X \rangle 0$

$Lim \frac{|X|}{X} \frac{+1}{-1} = -1$

$\chi \langle 0$

$Lim \frac{|X|}{X}$ does not exist

$f(\chi)$ discontinuous

Example 3: check the continuity of the function $\chi = 3$

$$f(\chi) = \begin{cases} \chi - 2 & \chi \neq 3 \\ 1 & \chi = 3 \end{cases}$$

SOL

f (3) = 1

$Lim(\chi - 2) = 3 - 2 = 1$

$\chi \to 3$

$f(3) = Lim\, f(\chi)$

$\chi \to 3$

The function continuous at $\chi = 3$

Problems

Q1// find Domain, range and sketch each of the following:

1 - $y = \chi^2$

2 - $y = \sqrt{X}$

3 - $y = 1\chi 1$

4- $y = |\chi + 2|$

5 - $y = \frac{|\chi|}{\chi}$

6 - $y = \frac{1}{\chi}$

7 - $y = \frac{\chi+1}{\chi-1}$ 8 - $y = 2\,Sin\,\chi$

9 - $y = -2\,Sin\,\chi$ 10 - $y = 2 + Cos\chi$

Q2 // Evaluate each of the following limits:

1 - $\lim_{y\to 2} \frac{t+3}{t+2}$ 2 - $\lim_{\chi\to 1} \frac{\chi^2-1}{\chi-1}$

3 - $\lim_{y\to 2} \frac{y2+5y+b}{y+2}$ 4 - $\lim_{y\to 2} \frac{y^2-5y+6}{y-2}$

5 - $\lim_{\chi\to -3} \frac{\chi2+4\chi+3}{\chi+3}$ 6 - $\lim_{t\to\infty} \frac{t+1}{t^2+1}$

7 - $\lim_{t\to\infty} \frac{t^2-2t}{2t^2+5t-3}$ 8 - $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$

9 - $\lim_{\theta\to 0} \frac{Sin\,2\theta}{\theta}$ 10- $\lim_{\chi\to 0} \frac{Sin\chi}{3\chi}$

11 - $\lim_{\chi\to 0} \frac{Sin\,5\,\chi}{Sin\,3\,\chi}$ 12 - $\lim_{\chi\to\infty} \chi\,Sin\frac{1}{\chi}$

13 - $\lim_{\chi\to 0} \frac{Sin^2\chi}{\chi}$ 14 - $\lim_{\chi\to 0} \frac{Sin^2\chi}{2\chi^2+\chi}$

15 - $\lim_{\chi\to 0} \tan 2\,\chi\,Csc\,4\,\chi$

Chapter Three

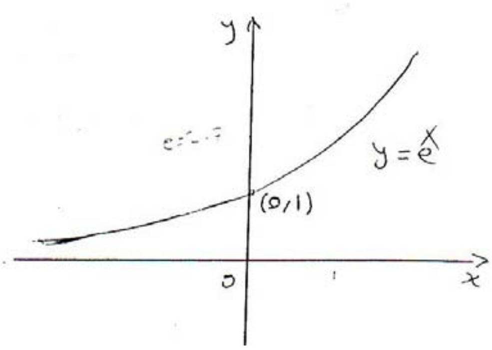
Special Function

1 - Exponential function

(i) $y = e^{X}$, $e = 2.7$

Domain: R

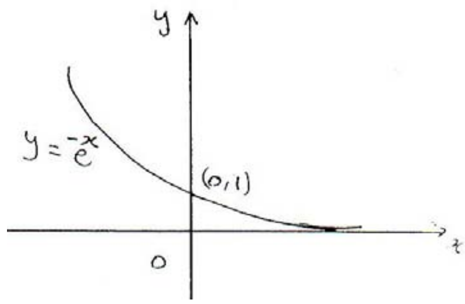
Range: R $(0,\infty)$



(ii) $y = e^{-X}$

Domain: R

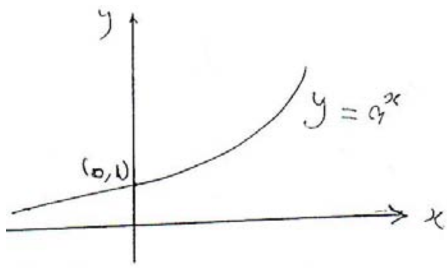
Range: R: $(0,\infty)$



(iii) $y = a^{X}$, $a \rangle 0$

Domain: R

Range: $(0,\infty)$



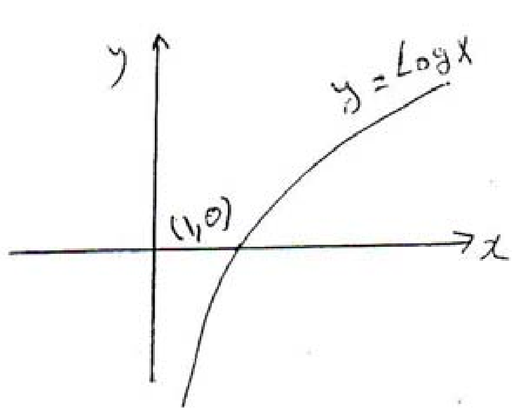
2 – Logarithmic Function:

(i) Common logarithmic function (Log X)

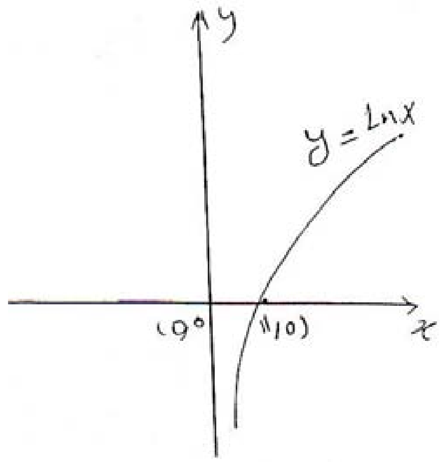
$$y = Log_{10} X \Leftrightarrow \chi = 10^{Y}$$

Domain: $(0,\infty)$

Range: R



(ii) Natural Logarithmic function (Ln x)



$y = Ln\,x \Leftrightarrow X = e^{X}$, $e = 2.7$

Domain: $(0, \infty)$

Range: R

Theorem:

If a and X are positive numbers and n is any rational number, then

(i) Ln1 = 0

(ii) Lne = 1

(iii) LnaX = Lna + Lnx

(iv) $Ln(\frac{X}{a}) = LnX - Lna$

(v) $Ln\,X^{n} = n\,LnX$.

Note:

(1) $e^{LnX} = X \quad \chi \rangle 0$

(2) $Ln\,e^{X} = \chi$

(3) $e^{X} e^{Y} = e^{X+y}$

(4) $\lim\limits_{\chi \to \infty} Ln\chi = \infty$

(5) $\lim\limits_{\chi \to -\infty} e^{X} = 0$

(6) $Ln\chi^{a} = a\,Ln\chi$

Hyperbolic Functions:

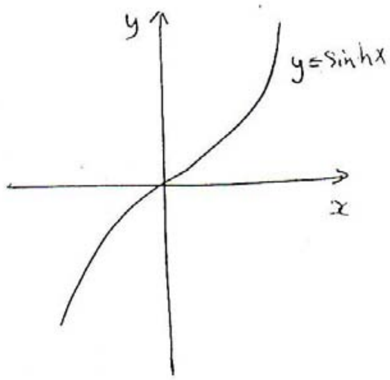
The Hyperbolic Functions are certain combinations of the exponential functions e^{x} and e^{-x} they are:

(i) Hyperbolic Sine (Sinh):

$$y = SinhX \, , SinhX = \frac{e^{x} - e^{-x}}{2}$$

Domain: R

Range: R

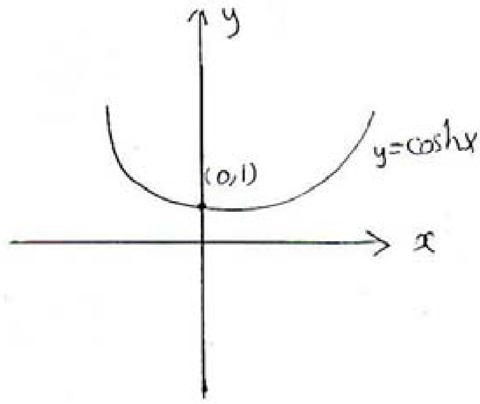


(ii) Hyperbolic Cosin (cosh)

$$y = Coshx \, , \, CoshX = \frac{e^{x} + e^{-x}}{2}$$

Domain: R

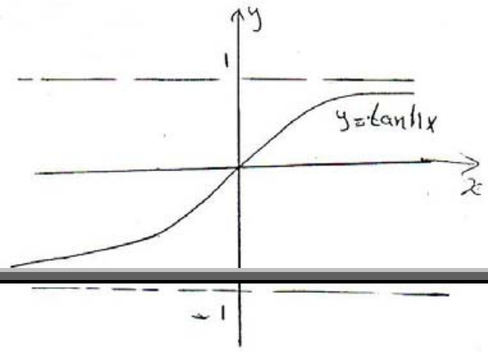
Range: $(1, \infty)$



(iii) Hyperbolic tangent (tanh)

$$y = \tanh x \quad , \quad \tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Domain: R



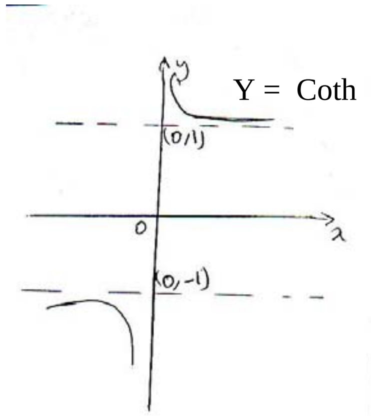
Range: (-1,1)

(iv) Hyperbolic cotangent (coth)

$$y = \coth\chi \ , \ \coth\chi = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

Domain: R - $\{0\}$

Range: $\{y : y\langle -1 \text{ or } \ y\rangle 1\}$

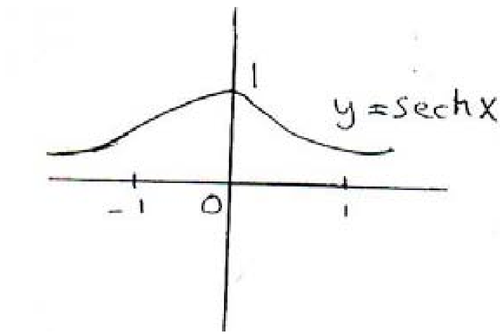


(v) Hyperbolic Secant (Sech)

$$y = Sech\chi \quad , \quad Sech\chi = \frac{2}{e^{x} + e^{-x}}$$

Domain: R

Range: (o,1)

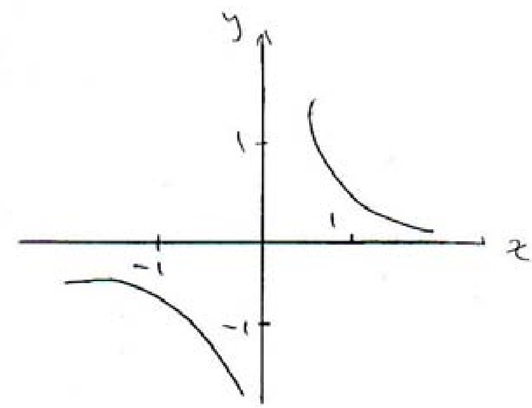


(vi) Hyperbolic cosecant (Csch)

$$y = Csch\chi \ , \ Csch\chi = \frac{2}{e^{x} - e^{-x}}$$

Domain: R - $\{0\}$

Range: R - $\{0\}$



Relationships among

Hyperbolic Function

1 - $Cosh^2\chi - Sinh2\chi = 1$

2 - $Sech^2\chi + \tanh^2\chi = 1$

3 - $Coth^2\chi - Csch^2\chi = 1$

Functions of negative arguments

1 - $Sinh(-\chi) = -Sinh\chi$

2 - $Cosh(-\chi) = Cosh\chi$

3 - $\tanh(-\chi) = -\tanh\chi$

4 - $Coth(-\chi) = -Coth\chi$

5 - $Sech(-\chi) = Sech\chi$

6 - $Csch(-\chi) = -Csh\chi$

Addition Formula:

1 - $Sinh(\chi \pm y) = Sinh\chi\ Coshy\ \mu\ Coshy\ Sinhy$

2 - $Cosh(\chi \pm y) = Cosh\chi\ Coshy \pm Sinh\chi\ Sinhy$

Double angle formula:

1 - $Sinh2\chi = 2Sinh\chi\ Cosh\chi$

2 - $$\begin{aligned} Cosh2\chi &= Cosh^2\chi + Sinh^2\chi \\ &= 1 + 2Sinh^2\chi \\ &= 2\ \ Cosh^2\chi - 1 \end{aligned}$$

Inverse Trigonometric Function

1 – Inverse Sine (Sin^{-1})

$$y = Sin^{-1}\chi = arc\ Sine \Leftrightarrow \chi = Siny.$$

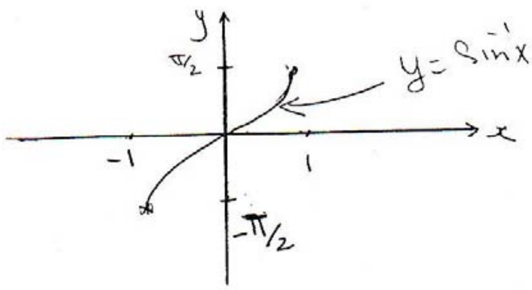
Y is the angle whose Sine is χ

Example:

$45^o = Sin^{-1}\frac{1}{\sqrt{2}}, \frac{\pi}{2} = Sin^{-1}1$

Domain: $[-1,1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ Principle values of y.



2 – Inverse Cosine (Cos^{-1})

$y = Cos^{-1}\chi = arc\,Cos\chi \to \chi = Cosy$

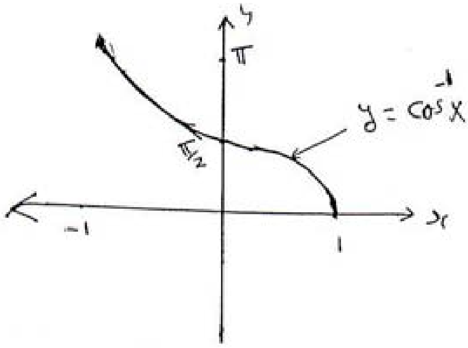
y is the angle whose cosin is χ

Example:

$30^o = Cos-1\sqrt{3/2} \to \frac{\sqrt{3}}{2} = Cos\,30^o$

Domain: $[-1,1]$

Range: $[0,\pi]$



3 – Inverse of tangent (tan $^{-1}$)

$y = \tan-1\chi = arc\tan\chi \Leftrightarrow \chi = \tan y$

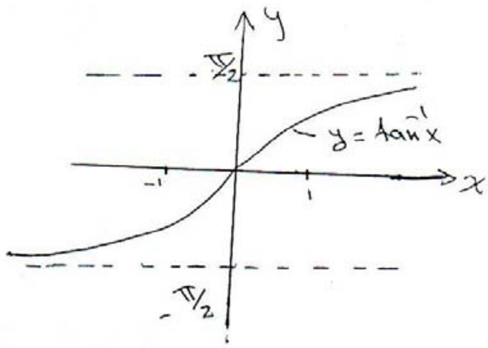
y = is the angle whose cosin is χ

Example:

$45^o = \tan^{-1}1 \Leftrightarrow 1 = \tan 45$

Domain: R

Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$

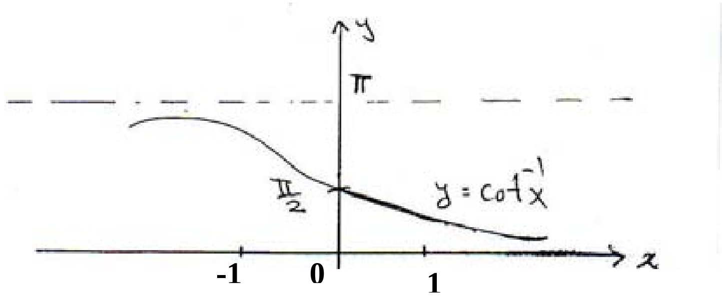


4 – Inverse of cotangent (Cot $^{-1}$)

$$y = Cot^{-1}\chi = arc\,Cot\,\chi \Leftrightarrow \chi = Coty$$

Domain: R

Range: $(0\ ,\ \pi)$

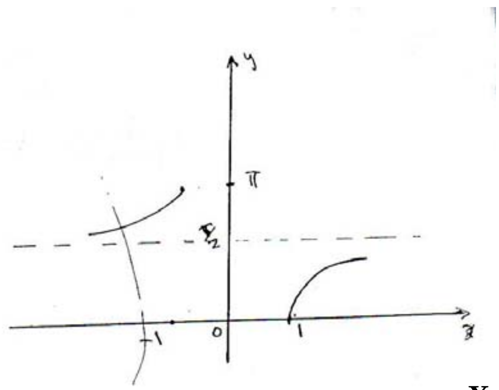


5 – Inverse secant (Sec $^{-1}$)

$$y = Sec^{-1}\chi = arc\,Sec\chi \rightarrow\ \chi = Secy$$

Domain: $(-\infty\ ,\ -1] \cup [1, +\infty)$

Range: $\left[0\ ,\ \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}\ \pi\right]$



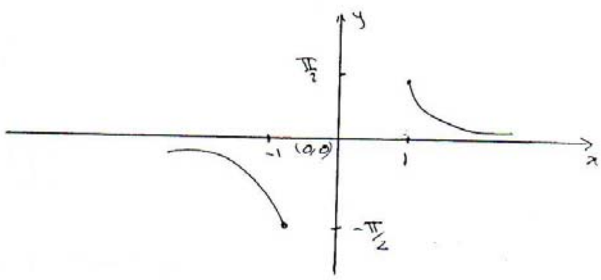
X

6 – Inverse Csc (Csc -1)

$$y = Csc^{-1}\chi = arc\,Csc\chi \Leftrightarrow X = Cscy$$

Domain: $(-\infty\ ,\ -1] \cup [1\ ,\ \infty)$

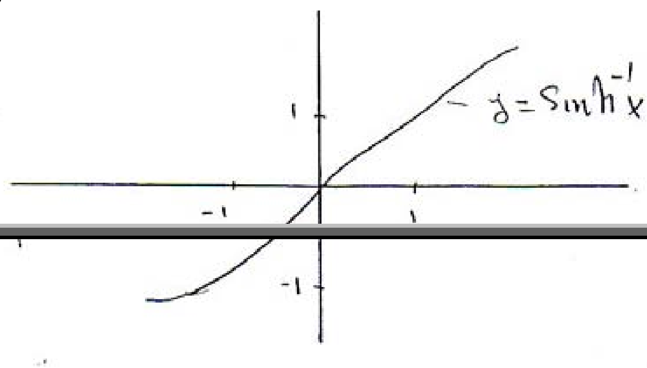
Range: $\left[-\frac{\pi}{2}\ ,\ 0\right) \cup \left(0\ ,\ \frac{\pi}{2}\right]$



Inverse hyperbolic Functions

1 – Inverse hyperbolic sine (Sinh -1)

$$y = Sinh^{-1}\chi \Leftrightarrow \chi = Sinhy$$



Domain: R

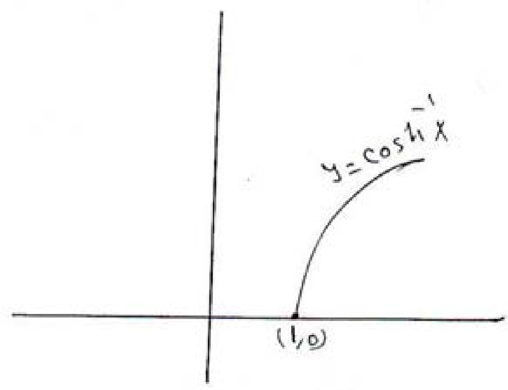
Range:R

2 - Inverse hyperbolic cosin (Cosh $^{-1}$)

$y = Cosh-1\chi \Leftrightarrow \; = Coshy$

Domain: $[1 , \infty)$

Range: $[0 , \infty)$

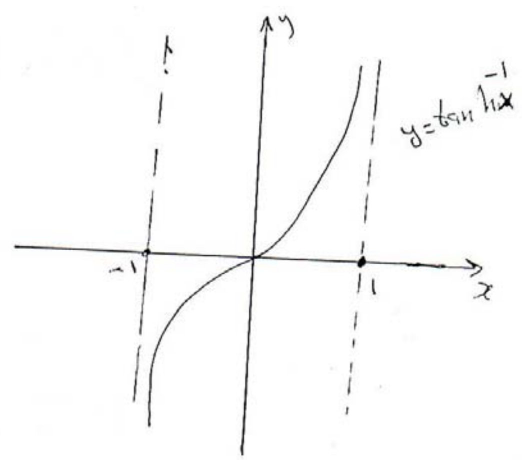


3 – Inverse hyperbolic tangent (tanh -1)

$y = \tanh-1\chi \Leftrightarrow \chi = \tanh y$

Domain: (-1, 1)

Range: R

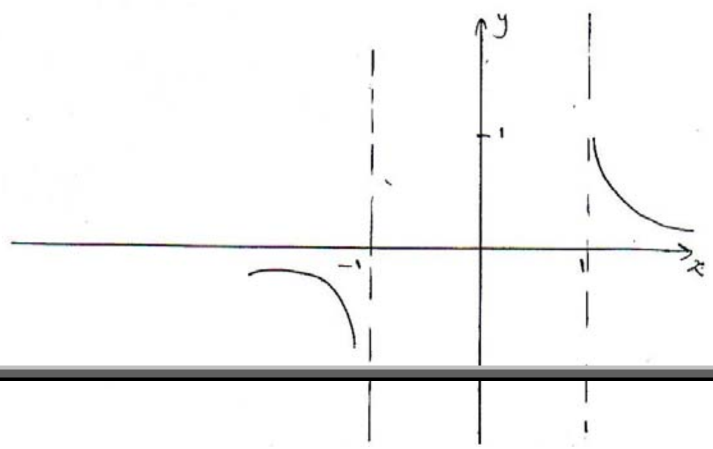


4 – Inverse hyperbolic cotangent (Coth $^{-1}$)

$y = Coth-1\chi \Leftrightarrow \chi = Cothy$

Domain: $\{X \rangle 1 \Upsilon X \langle -1\}$

Range: R / $\{0\}$

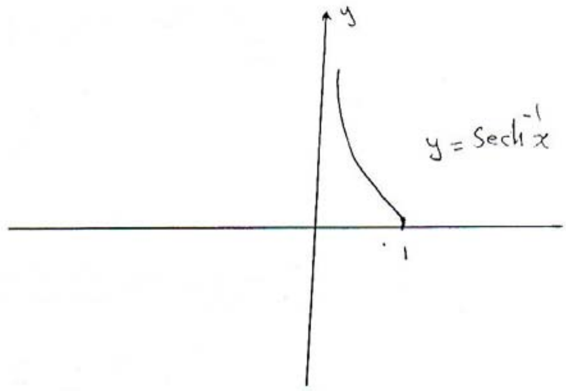


5 – Inverse hyperbolic secant (Sech -1)

$y = \sec h{-1}\chi \Leftrightarrow \chi = \sec hy$

Domain: $(0\ ,1]$

Range: $y \geq 0$



6 – Inverse hyperbolic cosecant (Csch -1)

$y = Csch{-1}\chi \Leftrightarrow \chi = Cschy$

Logarithmic Form of Inverse hyperbolic Functions

Theorem: the following relationships hold for all χ in the domain of the stated inverse hyperbolic.

Functions:

1 - $Sinh^{-1} = Ln(\chi + \sqrt{\chi 2 + 1})$

2 - $Cosh{-1}\chi = Ln\ (\chi + \sqrt{\chi 2 - 1})$

3 - $\tanh{-1}\chi = \frac{1}{2} \quad Ln\ (\frac{1+\chi)}{(1-\chi)})$

4 - $Coth{-1}\chi = \frac{1}{2}\ Ln\ \frac{(\chi+1)}{(\chi-1)}$

5 - $Sech{-1}\chi = Ln\ (\frac{1+\sqrt{1-\chi^2}}{\chi})$

6 - $Csch{-1}\chi = Ln\ (\frac{1}{\chi} + \frac{\sqrt{1+\chi^2}}{|\chi|})$

Prove that:

* $Sinh{-1}\chi = Ln(\chi + \sqrt{\chi 2 + 1})$

Sol

Let $y = Sinh{-1}\chi$

* $\chi = Sinhy$ Since $Sinhy = \frac{\overset{y}{e} - \overset{-y}{e}}{z}$

$X = \frac{\overset{y}{e} - \overset{-y}{e}}{z} \rightarrow\ 2X = \overset{y}{e} - \overset{-y}{e}$

$\left(\overset{y}{e} - z\chi - \overset{-y}{e} = 0\right)\overset{y}{e} \rightarrow \overset{2y}{e} - 2\chi\overset{y}{e} - 1 = 0$

$\overset{y}{e} = \frac{2\chi \pm \sqrt{4\chi^2 + 4}}{2} = \chi \pm \sqrt{\chi^2 + 1}$

Since $\overset{y}{e} \rangle\ 0$ then

$\overset{y}{e} = \chi + \sqrt{\chi^2 + 1} \rightarrow y = Ln(\chi + \sqrt{\chi 2 + 1})$ or

$Sinh{-1}\chi = Ln(\chi + \sqrt{\chi 2 + 1})$

Example:

$Sinh^{-1}\ 1 = Ln(1 + \sqrt{1+1})$

$= Ln(1+\sqrt{2})$

$= 0.88$

$* \ \tanh-1\chi = \frac{1}{2}\,Ln(\frac{1-\chi}{1+\chi})$

Sol

Let $y = \tanh-1\chi$

$$\therefore \chi = \tanh y = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} \quad \rightarrow \left[(e^{2y}+1)\ \chi = e^{y} - e^{-y}\right] e^{y}$$

$$(e^{2y}+1)\ \chi = e^{2y} - 1 \rightarrow \chi\, e^{2y} + \chi = e^{2y} - 1$$

$$\chi + 1 = e^{2y} - \chi\, e^{2y} \rightarrow e^{2y}\,(1-\chi) = 1+\chi$$

$$e^{2y} = \frac{1+\chi}{1-\chi} \rightarrow 2y = \frac{1}{2}\,Ln(\frac{1+\chi}{1-\chi})$$

$$\tanh-1 = \frac{1}{2}\,Ln(\frac{1+\chi}{1-\chi})$$

Example:

$$\tanh-1(\frac{1}{2}) = \frac{1}{2}\,Ln(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}) = \frac{1}{2}\,Ln3 = 0.5493$$

$$= 0.55$$

Problems:

Q1: find domain, range and sketch each of following:

1 - $y = Sin-1\chi \quad , \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

2 - $y = Cot-1\chi \quad , \quad 0 \langle\ y \ \langle\ \pi$

3 - $y = \tan-1\chi \quad , \quad -\frac{\pi}{2} \langle\ y \ \langle\ \frac{\pi}{2}$

4 - $y = \csc - 1\chi \quad , \quad -\frac{\pi}{2} \langle\; y \;\langle\; \frac{\pi}{2}$

5 - $y = Cot - 1\chi \quad , \quad \partial \;\langle\; y \;\langle\; \pi$

6 - $y = y = Sec - 1\chi, \partial \le y \le \pi$

7 - $y = 3Ln\chi$

8 - $y = -3Ln\chi$

9 - $y = 2\,e^{3X}$

10 - $y = 2\,e^{-3X}$

11 - $y = Sinh^{-1}\chi$

12 - $y = Cosh - 1\chi$

13 - $y = Cosh - 1\chi$

14 - $y = \tanh^{-1}\chi$

15 - $y = Coth^{-1}\chi$

16 - $y = Coth^{-1}\chi$

17 - $y = Sech^{-1}\chi$

18 - $y = \csc h^{-1}\chi$

Q2: Prove that:

1 - $Sinh - 1\chi = Ln(\chi + \sqrt{\chi^2 + 1}) \quad , \quad -\infty \langle\; \chi \;\langle\; \infty$

2 - $Coch - 1\chi = Ln(\chi + \sqrt{\chi^2 - 1}) \quad , \quad \chi \ge 1$

3 - $\tanh - 1\chi = \frac{1}{2} Ln(\frac{1+\chi}{1-\chi}) \quad , \quad |\chi| \;\langle\; 1$

4 - $Coth - 1\chi = \frac{1}{2} Ln(\frac{\chi+1}{\chi-1}) \quad , \quad |\chi| \;\rangle\; 1$

5 - $Sech - 1\chi = Ln(\frac{1+\sqrt{1-\chi^2}}{\chi}) \quad , \quad 0 \langle\; \chi \le 1$

6 - $Csch^{-1}\chi = Ln(\frac{1}{\chi} + \frac{\sqrt{1+\chi^2}}{|\chi|}) \quad , \chi \neq 0$

Q3// Discuss the Continuity of the following functions at the given points:

1 - $f(\chi) = |\chi + 4| \quad at \quad \chi = -4$

2 - $f(\chi) = \begin{cases} \sqrt{1-\chi^2} & if \ 0 \leq \chi < 1 \\ 1 & if \ 1 \leq \chi < 2 \\ 2 & if \ \chi = 2 \end{cases}$

$at \quad \chi = 0$

3 - $f(\chi) = \frac{|\chi|}{\chi} \ at \ \chi = 0$

4 - $f(\chi) = \begin{cases} \frac{1 - Cos\chi}{Sin2\chi} & for \ \chi \neq 0 \\ \frac{1}{2} & for \ \chi = 0 \end{cases}$

$at \ \chi = 0$

5 - $f(\chi) = \begin{cases} \chi - 2 & for \ \chi \neq 3 \\ 1 & for \ \chi = 3 \end{cases}$

$at \ \chi = 3$

6 - $f(\chi) = \begin{cases} \frac{\chi^3 - 8}{\chi 2 - 4} & for \ \chi \neq 2 \\ 0 & for \ \chi = 2 \end{cases}$

$at \quad \chi = 2$

7 - $f(\chi) = \begin{cases} Sin\,\pi\chi & 0 < \chi < 1 \\ Lnx & 1 < \chi < 2 \end{cases}$

$at \ \chi = 1$

8 - $f(\chi) = \dfrac{\chi - |\chi|}{\chi}$ $at\ \chi = 2$

9 - $f(\chi) = \begin{cases} \dfrac{|\chi - 3|}{\chi - 3} & for\ \chi \neq 3 \\ 0 & for\ \chi = 3 \end{cases}$ $at\ \ \chi = 3$

Q4 // Simplify each of the following:

1 - e^{LnX}

2 - $Ln\ (e^{X})$

3 - $e^{-Ln\,(X\,2)}$

4 - $Ln\ (\ e^{-X\,2}\ )$

5 - $Ln\ (e^{\frac{1}{X}})$

6 - $Ln\ (e^{\frac{1}{X}})$

7 - $e^{Ln(\frac{1}{X})}$

8 - $e^{-Ln(\frac{1}{X})}$

9 - $e^{Ln\,2\ +\ LnX}$

10 - e^{2LnX}

11 - $Ln\ (\ e^{X\ -\ X\,2}\ )$

12 - $Ln\ (\chi^{2}\ ,\ e^{-2X}\ )$

13 - $e^{X} + LnX$

14 - $e^{LnX} - 2Lny$

Chapter Four

The derivative

1 – Derivative of a function:

Let = $y = f(X)$ and let P(X1, y1) be fixed point on the curve, and Q $(X1 + \Delta X, y1 + \Delta y)$ is another point on the curve as see in the figure

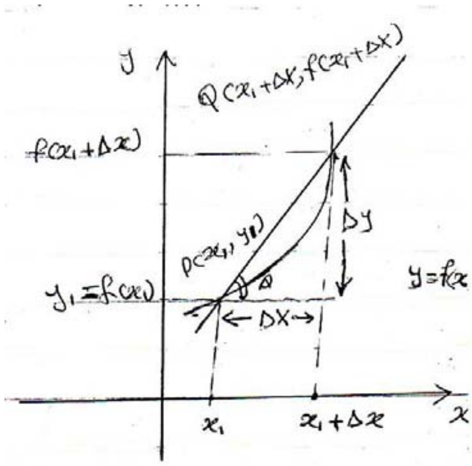
$y1 = f(X1)$, and

$y1 + \Delta y = f(X1 + \Delta X)$

$\Delta y = f(X1 + \Delta X) - y1$

Divided by ΔX

$$\frac{\Delta y}{\Delta X} = \frac{f(X_1 + \Delta X) - f(X_1)}{\Delta X}$$



The

slope of the curve $f(X)$ is

M = $\tan\phi = \frac{\Delta y}{\Delta X}$

$$\therefore M = \frac{f(X + \Delta X) - f(X)}{\Delta X}$$

We define the limit may exist for some value of X_1 .

At each point χ_1 where limit does exist, then f is said to have a derivative or to be differentiable.

Rules of Derivations:

$y = f(X) = C$ C constant

$y^1 = f^1(X) = \frac{dy}{dX} = 0$

$- f(X) = X^n$

$f^1(X) = nX^{n-1}$

$- f(X) = CnX^{n-1}$

$- f(X) = U \pm V$ n Positive integer

$f^1(X) = \frac{dy}{dX} = \frac{du}{dx} + \frac{dv}{dX}$

$- f(X) = UV$

$f^1(X) = U\frac{dv}{dX} + V\frac{du}{dX}$

$- f(X) = \frac{u}{v}$

$f^1(X) = \frac{vu^1 - uv^1}{V^2}$, $Where\ U^1 = \frac{du}{dX}$

$- f(X) = [U]^n$

$f^1(X) = n[U]^{n-1}\frac{du}{dX}$

$- f(X) = e^u$

$f^1(X) = e^u \frac{du}{dX}$

$- f(X) = C^U$ U C $Constant$

$f^1(X) = C^U . LnC . \frac{du}{dx}$

Derivative of trigonometric functions:

1) $(\sin u)' = \cos u\, du$
2) $(\cos u)' = -\sin u\, du$
3) $(\tan u)' = \sec^2 u\, du$
4) $(\cot u)' = -\csc^2 u\, du$
5) $(\sec u)' = \sec u \tan u\, du$
6) $(\csc u)' = -\csc u \cot u\, du$

Derivative of hyperbolic functions:

1) $\sinh u = \cosh u\, du$
2) $\cosh u = \sinh u\, du$
3) $\tanh u = \text{sech}^2 u\, du$
4) $\cot u = -\text{csch}^2 u\, du$
5) $\text{sech}\, u = -\text{sech}\, u \tanh u\, du$
6) $\text{csch}\, u = -\text{csch}\, u \coth u\, du$

derivative of the inverse trigonometric functions:

1) $(\sin^{-1} u)' = du/(1-u^2)^{1/2}$
2) $(\cos^{-1} u)' = -du/(1-u^2)^{1/2}$
3) $(\tan^{-1} u)' = du/1+u^2$
4) $(\cot^{-1} u)' = -du/1+u^2$
5) $(\sec^{-1} u)' = du/u(u^2-1)^{1/2}$
6) $(\csc^{-1} u)' = -du/u(u^2-1)^{1/2}$

derivative of the inverse of hyperbolic functions:

1) $(\sinh^{-1} u)' = du/(1+u^2)^{1/2}$
2) $(\cosh^{-1} u)' = du/(u^2-1)^{1/2}$
3) $(\coth^{-1} u)' = du/1-u^2$ if $|u|>1$
4) $(\tanh^{-1} u) = du/1-u^2$ if $|u|<1$

5) $(\text{sech}^{-1}u)' = -\, du/\, u(1-u^2)^{1/2}$

6) $(\text{csch}^{-1}u)' = -\, du/\, u(1+u^2)^{1/2}$

ex: find y' of

(1) $y = [\ln(3x+1)]^3$ (2) $y = 4^x$

Sol:

(1) $y' = 3[\ln(3x+1)]^2\, [3/(3x+1)] = 9[\ln(3x+1)]^2/(3x+1)$

(2) $y' = 4^x \ln 4$

{Applications of derivative}

Velocity and acceleration

Ex: find velocity and acceleration at time t to a moving body as

$S= 2t^3 - 5t^2 + 4t - 3$.

Sol:

$V= ds/dt = 6t^2 - 10t + 4$

$A= dv/dt = 12t-10$

<u>Theorem:</u>

Prove that:

$D(\sin^{-1}u) = 1/(1-u^2)^{1/2}\, (du/dx)$

<u>Proof</u>

Let $y = \sin^{-1} \rightarrow \sin y = u$ $u=[-1,1] \rightarrow y = [-\Pi/2 , \Pi/2]$

$\text{Cos } y\, dy/dx = du/dx \rightarrow dy/dx = 1/\cos y\, du/dx$

Since $\cos^2 y + \sin^2 y = 1$ this implies that

$\text{Cos } y = (1 - \sin^2 y)^{1/2} \rightarrow \text{Cos } y = \pm (1 - u^2)^{1/2}$

Cos y is positive between $-\Pi/2$ and $\Pi/2$

$Dy/dx = 1/(1-u^2)^{1/2}\, (du/dx) = D(\sin^{-1}u)$

Ex: find dy/dx for the following functions:

(1) $y = \tan(3x^2)$

(2) $y = x \sin^{-1} x + (1 - x^2)^{1/2}$

(3) $y = \cosh^{-1}(\sec x)$

sol:

(1) $y' = \sec^2(3x^2)\, 6x = 6x \sec^2(3x^2)$

(2) $y' = x/(1 - x^2)^{1/2} + \sin^{-1} x - x/(1 - x^2)^{1/2} = \sin^{-1} x$

(3) $y' = [1/(\sec^2 x - 1)^{1/2}] \sec x \tan x = \sec x \tan x/(\sec^2 x - 1)^{1/2}$

<u>Implicit relations:</u>

Ex: find dy/dx if

$x^5 + 4x\, y^3 - 3y^5 = 2$

sol:

$5x^4 + 4x\, 3y^2(dy/dx) + 4y^3 - 15y^4(dy/dx) = 0$

$(12x\, y^2 - 15y^4)\, dy/dx = - 5x^4 - 4y^3$

$dy/dx = (- 5x^4 - 4y^3)/(12x\, y^2 - 15y^4)$

<u>Chain Rule</u>

1- If y = f (x), and x = x (t), then

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\ \frac{\partial x}{\partial t}$$

2- If y = f (t), and x = x (t), then

$$\frac{\partial y}{\partial x} = \frac{\frac{\partial y}{\partial t}}{\frac{\partial x}{\partial t}}$$

Ex: find $\frac{dy}{dt}, \frac{dx}{dt}$ and $\frac{dy}{dx}$ of $x = 3t + 1$ and $y = t^2$

Sol:

$$\frac{dx}{dt} = 3.\ \frac{dy}{dt} = 2t,\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3}$$

<u>LHopital's Rule</u>

Let f and g be two functions which are differentiable in an open interval I containing the point c and let $g^{'}(x) \neq 0$. if

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}, \ \frac{\infty}{\infty}, \ 0.\infty, \ \infty.\infty, \ \infty.\infty, \ \frac{\infty}{0}, \infty^{0}, \text{ then}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^{'}(x)}{g^{'}(x)}.$$

Ex: evaluate (1) $\lim_{x \to 0} \frac{x^2 - \sin x}{x^2}$, (2) $\lim_{x \to 0} x \ln x$

Sol:

(1) $\lim_{x \to 0} \frac{x^2 - \sin x}{x^2} = \frac{0-0}{0} = \frac{0}{0}$

$$\lim_{x \to 0} \frac{x^2 - \sin x}{x^2} = \lim_{x \to 0} \frac{2x - \cos x}{2x} = \frac{-1}{0} = \infty$$

(2) $\lim_{x \to 0} x \ln x = 0.\infty$

$$\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$$

$$\lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0} x = 0$$

Series

(Power series): If $\{a_n\}$ is a sequence of constants, the expression:

$$a_0 + a_1 + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

is called power series in x.

(Taylor's series): If a function f can be represented by a power series in (x-b) called Taylor's series and has the form:

$$f(x) = f(b) + f^{'}(b)(x-b) + \frac{f^{''}(b)(x-b)^2}{2!} + ... + \frac{f^{n}(b)(x-b)^n}{n!} + ...$$

Example:

Find Taylor series expansion of cos x about a point a=2π

Sol:

f(x) = cos x

f(2π) = cos (2π) = 1

$$f^{'}(x) = -\sin x, \ \ f^{'}(2\pi) = -\sin 2\pi = 0$$

$$f^{''}(x) = -\cos x, \ \ f^{''}(2\pi) = -\cos 2\pi = -1$$

$$f^{'''}(x) = \sin x, \ \ f^{'''}(2\pi) = \sin 2\pi = 0$$

$$f^{iv}(x) = \cos x, \ \ f^{iv}(2\pi) = \cos 2\pi = 1$$

$$\cos x = 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \frac{(x-2\pi)^6}{6!} + ...$$

(Maclaurin series): when b = 0, Taylor series called Maclaurin series.

Example:

Find Maclaurin series for the function f(x) = e^x

Sol:

f(x) = $e^x \rightarrow$ f(0) = e^0 = 1

$f^{'}(x) = e^x \rightarrow f^{'}(0) = e^0 = 1$

$f^{''}(x) = e^x \rightarrow f^{''}(0) = e^0 = 1$

$f^{'''}(x) = e^x \rightarrow f^{'''}(0) = e^0 = 1$

$e^x = 1 + x + (x^2/2!) + (x^3/3!) + ...$

Chapter five

(INTEGRALS)

The process of finding the function whose derivative is given is called integration, it's the inverse of differentiation.

Definition:(indefinite integral)

A function $y=F(x)$ is called a solution of $dy/dx=f(x)$ if $dF(x)/dx=f(x)$.

We say that $F(x)$ is an integral of $f(x)$ with respect to x and $F(x) + c$ is also an integral of $f(x)$ with a constant c s.t

$D(F(x) +c)=f(x)$.

Formulas of Integration:

1) $\int dx=x + c$.
2) $\int a\, dx=a \int dx$
3) $\int(du \pm dv)= \int du \pm \int dv$.
4) $\int x^n\, dx=(x^{n+1}/n+1)+ c$
5) $\int (u)^n\, du=(u^{n+1}/n+1)+ c$
6) $\int e^u\, du= e^u+ c$
7) $\int a^u\, du=(a^u/\ln a) + c$
8) $\int du/u= \ln u + c$.

Example1:

Solve the differential equation: $dy/dx=3x^2$.

Sol:

$dy=3x^2\, dx$

since $d(x^3)=3x^2\, dx$, then we have:

$\int dy= \int 3x^2\, dx= \int d(x^3)\, dx$

$y= x^3+ c$.

9 methods for finding integrals:

1"Integral of trigonometric functions":

1) $\int \cos u\, du = \sin u + c$
2) $\int \sin u\, du = -\cos u + c$
3) $\int \sec^2 u\, du = \tan u + c$
4) $\int \csc^2 u\, du = -\cot u + c$
5) $\int \sec u \tan u\, du = \sec u + c$
6) $\int \csc u \cot u\, du = -\csc u + c$

2"Integral of hyperbolic functions":

1) $\cosh u\, du = \sinh u + c$
2) $\int \sinh u\, du = \cosh u + c$
3) $\int \text{sech}^2 u\, du = \tanh u + c$
4) $\int \text{csch}^2 u\, du = -\cot u + c$
5) $\int \text{sech}\, u \tanh u\, du = -\text{sech}\, u + c$
6) $\int \text{csch}\, u \coth u\, du = -\text{csch}\, u + c$

Integral of the inverse trigonometric functions:

1) $\int du/(1-u^2)^{1/2} = \{\sin^{-1}u + c \quad \text{or} \quad -\cos^{-1}u + c\}$
2) $\int du/1+u^2 = \{\tan^{-1}u + c \quad \text{or} \quad -\cot^{-1}u + c\}$
3) $\int du/\, u(u^2-1)^{1/2} = \sec^{-1}u + c \quad \text{or} \quad -\csc^{-1}u + c\}$

Integral of the inverse of hyperbolic functions:

1) $\int du/(1+ u^2)^{1/2} = \sinh^{-1}u + c$
2) $\int du/\, (u^2-1)^{1/2} = \cosh^{-1}u + c$
3) $\int du/1-u^2 = \tanh^{-1}u + c$ if $|u|<1$ and $\int du/1-u^2 = \coth^{-1} + c$ if $|u|>1$
4) $\int du/\, u(1-u^2)^{1/2} = -\text{sech}^{-1}u + c$
5) $\int du/\, u(1+u^2)^{1/2} = -\text{csch}^{-1}u + c$

ex:

evaluate:

1) $\int (5x^4-6x^2+2/ x^2)dx$

2) $\int \cos 2x\, dx$

3) $\int \cos^2 x\, dx$

sol:

1) $\int (5x^4-6x^2+2/ x^2)dx = 5\int x^4dx-6\int x^2dx+2\int x^{-2}dx$

$X^5 -2x^3-2/x+ c$

2) $\int \cos 2x\, dx = \sin 2x/2 + c$

3) $\int \cos^2 x\, dx = 1/2 \int(1+ \cos 2x)\, dx = 1/2[\int dx + \int \cos 2x\, dx]$

$1/2[x + \sin 2x/2] + c = 1/2x + 1/4 \sin 2x + c$

3- "Integration by parts"

Let u and v be functions of x and d(uv) = u dv + v du

By integration both sides of this equation (w.r.t x)

$\int d(uv)= \int u\, dv + \int v\, du$ this implies $(uv)= \int u\, dv + \int v\, du$

$\int u\, dv= (uv) - \int v\, du$

ex: find $\int x e^x dx$

sol:

let $u=x \rightarrow du = dx$ and let $dv = e^x dx$

from $\int u\, dv= (uv) - \int v\, du \rightarrow \int x e^x dx= x e^x- e^x +c$

4- "Integrals involving $(a^2 - u^2)^{1/2}$, $(a^2 + u^2)^{1/2}$, $(u^2 - a^2)^{1/2}$, $a^2 - u^2$, $a^2 + u^2$, $u^2 - a^2$ "

(A) $u= a \sin \Phi$ replaces $a^2 - u^2 = a^2 - a^2 \sin^2 \Phi = a^2(1- \sin^2 \Phi) = a^2\cos^2 \Phi$

(B) $u= a \tan \Phi$ replaces $a^2 + u^2 = a^2 + a^2 \tan^2 \Phi = a^2 \sec^2 \Phi$

(C) $u= a \sec \Phi$ replaces $u^2 - a^2 = a^2 \sec^2 \Phi - a^2 = a^2 \tan^2 \Phi$

Ex: find $\int dx/x^2(4 - x^2)^{1/2}$

Sol:

Let $x=2\sin\Phi \rightarrow dx = 2\cos\Phi\, d\Phi$

$\int dx/x^2(4 - x^2)^{1/2} = \int 2\cos\Phi\, d\Phi/4\sin^2\Phi(4- 4\sin^2\Phi)^{1/2} =$

$\int 2\cos\Phi\, d\Phi/4\sin^2\Phi(2\cos\Phi) = \int d\Phi/4\sin^2\Phi = 1/4\int \csc^2\Phi\, d\Phi = -1/4\cot\Phi + c$

now

from $x= 2\sin\Phi \rightarrow \sin\Phi = x/2 \rightarrow \cos\Phi = (1- x^2/4)^{1/2} = 1/2(4 - x^2)^{1/2}$

$-1/4\cot\Phi + c = (-1/4)(4 - x^2)^{1/2}/x$

5-" Integrals involving $ax^2 + bx + c$ "

First, We put the equation as $(ax^2 + bx) + c$.

Second, if $a \neq 1$, we take a as a mutable by the sides of the equation which has x, $a[x^2 + (b/a)x] + c$.

Third, put and sub to the equation $[(1/2)$ the number multiplied by $x]^2$,

$a[x^2 + (b/a)x + (1/4)(b/a)^2 - (1/4)(b/a)^2] + c$.

Fourth, rewrite the equation as $a[x^2 + (b/a)x + (1/4)(b/a)^2] + c - (1/4)(b^2/a)$

Last, the equation become $a[x+ (1/2) (b/a)]^2 + c - (1/4)(b^2/a)$ and suppose $u = x+ (1/2) (b/a)$ to become $a[u]^2 + c - (1/4)(b^2/a)$

Ex: Find $\int dx/(4x^2 + 4x + 2)$

Sol:

$4x^2 + 4x + 2 = (4x^2 + 4x) + 2 = 4(x^2 + x) + 2 =$

$4[x^2 + x +(1/4) -(1/4)] +2= 4[x^2 + x +(1/4)] +2 -1=$

$4[x+1/2]^2 +1$.

Let $u = x+1/2 \rightarrow 4[x+1/2]^2 +1 = 4u^2 +1$.

Since $u = x+1/2 \rightarrow x = u - (1/2) \rightarrow dx = du$

$\int dx/(4x^2 + 4x + 2) = \int du/(4u^2 +1) = 1/2 \int 2du/(4u^2 +1) =$

$1/2 \tan^{-1}2u = 1/2 \tan^{-1}2(x+1/2).$

<u>6-"method of partial fractions"</u>

If the integral of the form f(x)/g(x) s.t f(x) and g(x) are poly.

And degree of f(x)< degree of g(x) we can carry out two cases:

Case i

If all factor of g(x) are linear, by the following ex:

<u>Ex</u>: find $\int dx/ x^2 + x – 2$

<u>Sol</u>:

$1/ x^2 + x – 2 = 1/(x-1)(x+2) = A/(x-1) + B/(x + 2) =$

$[A(x + 2) \ + B/(x - 1)] /(x-1) (x + 2)$

$1= Ax +A2 +Bx –B = (A+B)x + (A2-B)$

$1=A2-B$

<u>$0= (A+B)$</u>

$3A=1 \rightarrow A=1/3$ put in eq.(2) $\rightarrow B =-1/3$

$\int dx/ x^2 + x – 2 = \int 1/(x-1)(x+2)dx = \int [A/(x-1) + B/(x + 2)]dx$

$= \int [(1/3)/(x-1) – (1/3)/(x + 2)]dx = 1/3\int /(x-1) – 1/3 \int /(x + 2)dx$

$= 1/3 \ln|x-1| -1/3 \ln|x+2|+c$

Case ii

If some of the factors of g(x) are quadratic, by the following ex:

Ex: find $\int (x^2 + x – 2)dx/ (3x^3- x^2 + 3x – 1)$

Sol:

$(x^2 + x – 2) / (3x^3- x^2 + 3x – 1) = (x^2 + x – 2)/ x^2(3x – 1) + (3x – 1)$

$= (x^2 + x – 2)/ (3x – 1) (x^2+1) = [A/(3x – 1)] +[(Bx + C)/ (x^2+1)]$

$= [A(x^2+1)+ (Bx + C) (3x – 1)]/ (3x – 1) (x^2+1)$

$x^2 + x – 2 = A(x^2+1)+ (Bx + C) (3x – 1)$

$x^2 + x – 2 = (A + 3B) x^2 + (B + 3C) x + (A-C)$

$A + 3B= 1$

$B + 3C = 1$

<u>$A - C = -2$</u>

$A = -7/5$, $B= 4/5$, $C = 3/5$

$(x^2 + x - 2)/ (3x - 1) (x^2+1) = (-7/5)/(3x - 1) +[(4/5)x + (3/5)]/ (x^2+1)$

And

$\int (x^2 + x - 2)dx/ (3x - 1) (x^2+1) = (-7/5)\int dx/(3x - 1) +$

$(4/5) \int x\, dx/(x^2+1) + (3/5) \int dx/ (x^2+1)$

$=-(7/15) \ln |3x-1| + (2/5) \ln | x^2+1| + 3/5 \tan^{-1} x$

<u>7-"further substitutions"</u>

Some integrals involving fractional powers of the variable x may be simplified by substitution $x = u^n$ where n is the least common multiple of the denominators of the exponents.

<u>Ex:</u>

$I=\int (x)^{1/2}dx/1+(x)^{1/3}$

<u>sol:</u>

let $x=u^6 \rightarrow dx = 6 u^5 du$

$I= \int (u^6)^{1/2}(6 u^5) du/(1+(u^6)^{1/3}) = 6 \int u^3 u^5du/1+u^2= 6 \int u^8du/1+u^2$

By long division

$u^8/1+u^2 = u^6 - u^4 + u^2 -1 +(1/1+ u^2)$

$I= 6\int u^8du/1+u^2 = 6 \int [u^6 - u^4 + u^2 -1 +(1/1+ u^2)] du$

$= (6/7) u^7 - (6/5) u^5 + 2 u^3 -6u + 6 \tan^{-1}u+c$

$= (6/7) x^{7/6} - (6/5) x^{5/6} + 2 x^{1/2} -6x^{1/6} + 6 \tan^{-1}(x^{1/6})+c$

<u>8-"rational functions of sin x and cos x"</u>

If the integral that is rational function of sin x or cos x or both, can be changed as following:

Let $z = \tan(x/2)$

$x/2 = \tan^{-1}z \rightarrow x = 2\tan^{-1}z \rightarrow dx = 2dz/(1+ z^2)$

$\cos(x/2) = 1/(1+ z^2)^{1/2}$, $\sin(x/2) = z/(1+ z^2)^{1/2}$

$\sin x = 2\sin(x/2)\cos(x/2) = 2z/(1+z^2)$

$\cos x = \cos^2(x/2) - \sin^2(x/2) = (1-z^2)/(1+z^2)$ from $[\cos(x/2 + x/2)]$

ex: find $I = \int dx/(1 - \sin x + \cos x)$

sol:

$$I = \int \frac{2dz/(1+z^2)}{(1 - [2z/(1+z^2)] + [(1-z^2)/(1+z^2)])}$$

$$= \int \frac{2dz/(1+z^2)}{(1+z^2 - 2z + 1 - z^2)/(1+z^2)}$$

$= \int 2dz/(2-2z) = \int dz/(1-z) = -\ln|1-z| + c = -\ln|1-\tan(x/2)| + c$

9-"evaluating integrals of the following types"

(A) $\sin(mx)\sin(nx) = (1/2)[\cos(m-n)x - \cos(m+n)x]$

(B) $\sin(mx)\cos(nx) = (1/2)[\sin(m-n)x + \sin(m+n)x]$

(C) $\cos(mx)\cos(nx) = (1/2)[\cos(m-n)x + \cos(m+n)x]$

Ex:

$\int 2\sin(4x)\sin(3x)\,dx = \int (2/2)[\cos(4-3)x - \cos(4+3)x]dx$

$= \int (\cos x - \cos 7x)dx = \sin x - (1/7)\sin 7x + c$

{definite integral}

The definite integral like indefinite integral but there is a limit to the integral like $\int_a^b f(x)\,dx = F(a) - F(b)$.

Ex: evaluate $\int_0^3 x^3\,dx$

Sol:

$(3)^4/4 - 0 = 81/4$

Applications of definite integral

{area under the curve}

Ex: find the area under sin x bdd by x= 0 and x = 2π and x-axis

Sol:

A= $\int_0^{2\pi}$ sin x dx = $\int_0^{\pi}$ sin x dx- $\int_{\pi}^{2\pi}$ sin x dx = (cos π-cos 0)+(cos 2π-cos π)= (-1-1)+(1-(-1))=4

{area between two curves}

Ex: find the area bdd by y= 2- x^2 and y = -x

Sol:

Y= 2- x^2= y = -x

2- x^2 = -x → x^2 –x-2=0 → (x-2)(x+1) = 0 → x=2, x= -1

A= $\int_{-1}^{2}[(2-x^2)-(-x)]dx= [2x-(x^3/3) + (x^2/2)]^2_{-1}$= 6.5

<u>Double integrals</u>

When the integral have two signals of integral to two parameters x and y called double integral, like $\int\int$ f(x,y) dx dy.

The benefit of like integrals is to find the volume of things.

Ex: find the volume of f(x,y) = x^2y limited by x=(1,3) and y =(1.2)

$\int_1^2 \int_1^3 x^2y\, dx\, dy = \int_1^2 [(x^3/3)y\, dy]^3_1 = \int_1^2 [(3^3/3)- (1^3/3)]y\, dy$

$= \int_1^2 [(27/3)- (1/3)]y\, dy = \int_1^2 (26/3)y\, dy = (26/3)\int_1^2 y\, dy$

$= [(26/3)(y^2/2)]^2_1 = [(13/3)\, y^2]^2_1$ = (13/3) 4- (13/3) =13